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b-CHROMATIC NUMBER OF LEXICOGRAPHIC PRODUCT OF SOME GRAPHS

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Abstract A $b-$coloring of a graph G is a coloring of the vertices of G such that each color class contains at least one vertex that has a neighbour in all other color classes. The $b-$chromatic number of a graph G, denoted by $\chi_b(G)$, is the largest integer k such that G admits a $b-$coloring with k colors. In this paper, we obtain the $b-$Chromatic number of lexicographic product of two graphs G and H, denoted by $G[H]$. First, we consider the graph $G[H]$, where G is the path graph, and H is the sunlet graph and wheel graph. Secondly, we consider G as the cycle graph and H as the wheel graph respectively. Finally, consider G and H are the wheel graphs.

1 Introduction

All graphs considered in this paper are non-trivial, simple and undirected. A $k-$coloring (we may refer to it simply as a coloring) of a graph $G = (V, E)$ is a function $c : V \to \{1, 2, \ldots, k\}$, such that $c(u) \neq c(v)$ for all $uv \in E(G)$. The color class c_i is the subset of vertices of G that are assigned to color i. The chromatic number of G, denoted $\chi(G)$, is the smallest integer k such that G admits a $k-$coloring. The problem of determining the chromatic number of a graph is widely studied [10],[14]. In particular, because of its many applications, since it corresponds to the fundamental problem of determining an optimal partition of a set of objects into classes according to some restriction. Problems of scheduling, frequency assignment [8] and register allocation [4],[5], besides of the finite element method, are naturally modelled by the coloring problem.

Given a coloring c, a vertex v is a $b-$vertex of color i, if $c(v) = i$ and v has at least one neighbour in every color class c_j, $j \neq i$. A $b-$coloring is a coloring such that each color class has a $b-$vertex. The $b-$chromatic number of a graph G, denoted $\chi_b(G)$, is the largest integer k such that G admits a $b-$coloring with k colors. A $b-$coloring may be obtained by the following heuristic that improves some given coloring of a graph G. One can start with any coloring c of G and, as long as possible, do the following: pick-up a color class of c with no $b-$vertices and recolor every vertex v in this class with some color that does not occur in its neighborhood. If c is not a $b-$coloring, this process produces a coloring c' (which is a $b-$coloring) better than c in terms of the number of used colors. Observe that an optimal vertex coloring is necessarily a $b-$coloring, and then the $b-$chromatic number is an upper bound for the chromatic number of a graph. Since it is very easy to obtain a $b-$coloring of a graph, and since any $b-$coloring provides an upper bound on the chromatic number [9], a natural application of the $b-$coloring is to evaluate the performance of any graph coloring heuristics. On the other hand, the concept of $b-$coloring was used in databases clustering [6] and in automatic recognition of documents [2]. The $b-$colorings were first defined in [7]. In that paper, Irving and Manlove prove that the problem of determining the $b-$chromatic number of a graph is NP-Hard. In fact, it is shown in [11] that deciding whether a graph admits a $b-$coloring with a given number of colors is an NP-complete problem, even for connected bipartite graphs.

2 Preliminaries

A trail is called a path if all its vertices are distinct. A closed trail whose origin and internal vertices are distinct is called a cycle. [1]

For any positive integer $n \geq 4$, the wheel graph W_n is the $n-$vertex graph obtained by joining a vertex v_1 to each of the $n-1$ vertices $\{w_1, w_2, \ldots w_{n-1}\}$ of the cycle graph C_{n-1} [12].

The $n-$sunlet graph is the graph on $2n$ vertices obtained by attaching n pendant edges to a cycle graph C_n and it is denoted by S_n [12].

Lexicographic product was first introduced by Felix Hausdorff in 1914. In graph theory, the Lexicographic Product $G[H]$ of graphs G and H is a graph such that the vertex set of $G{\cdot}H$ is the cartesian product $V(G) \times V(H)$ [13] and any two vertices (u, v) and (x, y) are adjacent in $G[H]$ if and only if either

- u is adjacent with x in G or
- u=x and v is adjacent with y in H.

The Lexicographic product is also called the composition [3].

3 Main Results

In this section, we obtain the $b-$Chromatic number of lexicographic product of two graphs G and H, denoted by $G\,[H]$. First, we consider the graph $G\,[H]$, where G is the path graph, and H is the sunlet graph and wheel graph. Secondly, we consider G as the cycle graph and H as the wheel graph respectively. Finally, consider G and H are the wheel graphs.

First we consider the graph G be the isomorphic to the path graph of order m vertices and H be the isomorphic to the sunlet graph of order n vertices. Let $V(G) = \{u_i : 1 \leq i \leq m\}$ and $V(H) = \{v_j : 1 \leq j \leq 2n\}$, where $v_j, (j = 1, 2, \ldots n)$ are the vertices of cycle taken in a cyclic order and $v_{n+j}, (j = 1, 2, \ldots n)$ are pendant vertices such that each $v_j v_{n+j}$ are the pendant edge. Let $V(G[H]) = \bigcup\limits_{i=1}^{m} \{x_{i,j} : 1 \leq j \leq 2n\}$, where $x_{i,j}$ are the vertices of $u_i v_j (1 \leq i \leq m, 1 \leq j \leq 2n)$.

Theorem 3.1. *The graph G be the isomorphic to the path graph of order m vertices and H be the isomorphic to the sunlet graph of order n vertices. Then the $b-$chromatic number of lexicographic product of $G[H]$ is 6.*

Proof. Define a mapping $\sigma : V(G[H]) \to N$ as follows:

Case 1: For n is odd

$$\begin{aligned}
\sigma(x_{2i-1,2j}) &= 3, \text{ for } 1 \leq i \leq \frac{m}{2}, 1 \leq j \leq n-1; \\
\sigma(x_{2i-1,2j+1}) &= 5, \text{ for } 1 \leq i \leq \frac{m}{2}, 1 \leq j \leq n-1; \\
\sigma(x_{2i-1,1}) &= \sigma(x_{2i-1,2n}) = 1, \text{ for } 1 \leq i \leq \frac{m+1}{2}; \\
\sigma(x_{2i,2j}) &= 4, \text{ for } 1 \leq i \leq \frac{m}{2}, 1 \leq j \leq n-1; \\
\sigma(x_{2i,2j+1}) &= 6, \text{ for } 1 \leq i \leq \frac{m}{2}, 1 \leq j \leq n-1; \\
\sigma(x_{2i,1}) &= \sigma(w_{2i,2n}) = 2, \text{ for } 1 \leq i \leq \frac{m-1}{2}.
\end{aligned}$$

Case 2: For n is even

$$\begin{aligned}
\sigma(x_{2i-1,2j}) &= 3, \text{ for } 1 \le i \le \frac{m}{2}, 1 \le j \le \frac{n}{2};\\
\sigma(x_{2i-1,2j+1}) &= 5, \text{ for } 1 \le i \le \frac{m}{2}, 1 \le j \le \frac{n}{2};\\
\sigma(x_{2i-1,1}) &= 1, \text{ for } 1 \le i \le \frac{m}{2};\\
\sigma(x_{2i,2j}) &= 4, \text{ for } 1 \le i \le \frac{m}{2}, 1 \le j \le \frac{n}{2};\\
\sigma(x_{2i,2j+1}) &= 6, \text{ for } 1 \le i \le \frac{m}{2}, 1 \le j \le \frac{n}{2};\\
\sigma(x_{2i,2}) &= 2, \text{ for } 1 \le i \le \frac{m}{2};\\
\sigma(x_{2i-1,2n}) &= 5, \text{ for } 1 \le i \le \frac{m+1}{2};\\
\sigma(x_{2i,2n}) &= 6, \text{ for } 1 \le i \le \frac{m-1}{2}.
\end{aligned}$$

By assumption, $\chi_b(G[H]) \ge 6$. Let us assume that $\chi_b(G[H])$ is greater than 6. As cycle C_n increases, the adjacency between any two vertex decreases and hence the color assigned to the corresponding vertices doesn't form anyone of the color class which contradict the definition of $b-$coloring which states that any two assigned color must exist at least once. So, $\chi_b(G[H]) \le 6$. But, the $b-$chromatic number of $\chi_b(G[H])$ is the largest integer. Therefore $\chi_b(G[H]) = 6$. □

We consider the graph G be the isomorphic to the path graph of order m vertices and H be the isomorphic to the wheel graph of order n vertices. Let $V(G) = \{u_i : 1 \le i \le m\}$ and $V(H) = \{v_1\} \cup \{v_j : 2 \le j \le n\}$, where v_j's are the vertices obtained by joining a vertex v_1 of the $n-1$ vertices and $\{v_2 \cdots v_n\}$ of the cycle graph. Let $V(G[H]) = \bigcup_{i=1}^{m} \{x_{i,j} : 1 \le j \le n\}$; where $x_{i,j}$ are the vertices of $u_i v_j (1 \le i \le m, \quad 1 \le j \le n)$.

Theorem 3.2. *The graph G be the isomorphic to the path graph of order m vertices and H be the isomorphic to the wheel graph of order n vertices.Then*

$$\chi_b(G[H]) = \begin{cases} 8, & for \quad n \ne 5 \\ 6, & for \quad n = 5 \end{cases}$$

Proof. Define a mapping $\sigma : V(G[H]) \to N$ as follows:

Case 1: For $n \ne 5$

Subcase 1: For m is even and n is even

$$\begin{aligned}
\sigma(x_{2i-1,j}) &= 2j-1, \text{ for } 1 \le i \le \frac{m}{2}, 1 \le j \le 4;\\
\sigma(x_{2i,j}) &= 2j, \text{ for } 1 \le i \le \frac{m}{2}, 1 \le j \le 4;\\
\sigma(x_{2i-1,2j-1}) &= 3, \text{ for } 3 \le j \le \frac{n}{2}, 1 \le i \le \frac{m}{2};\\
\sigma(x_{2i-1,2j}) &= 5, \text{ for } 3 \le j \le \frac{n}{2}, 1 \le i \le \frac{m}{2};\\
\sigma(x_{2i,2j-1}) &= 4, \text{ for } 3 \le j \le \frac{n}{2}, 1 \le i \le \frac{m}{2};\\
\sigma(x_{2i,2j}) &= 6, \text{ for } 3 \le j \le \frac{n}{2}, 1 \le i \le \frac{m}{2}.
\end{aligned}$$

Subcase 2: For m is odd & n is odd

$$\begin{aligned}
\sigma(x_{2i-1,j}) &= 2j-1, \text{ for } 1 \le i \le \frac{m+1}{2}, 1 \le j \le 4;\\
\sigma(x_{2i,j}) &= 2j, \text{ for } 1 \le i \le \frac{m-1}{2}, 1 \le j \le 4;\\
\sigma(x_{2i-1,3j-1}) &= 3, \text{ for } 2 \le j \le \frac{n-3}{2}, 1 \le i \le \frac{m+1}{2};\\
\sigma(x_{2i-1,3j}) &= 5, \text{ for } 2 \le j \le \frac{n-3}{2}, 1 \le i \le \frac{m+1}{2};\\
\sigma(x_{2i-1,3j+1}) &= 7, \text{ for } 2 \le j \le \frac{n-3}{2}, 1 \le i \le \frac{m+1}{2};\\
\sigma(x_{2i,3j-1}) &= 4, \text{ for } 2 \le j \le \frac{n-3}{2}, 1 \le i \le \frac{m-1}{2};\\
\sigma(x_{2i,3j}) &= 6, \text{ for } 2 \le j \le \frac{n-3}{2}, 1 \le i \le \frac{m-1}{2};\\
\sigma(x_{2i,3j+1}) &= 8, \text{ for } 2 \le j \le \frac{n-3}{2}, 1 \le i \le \frac{m-1}{2}.
\end{aligned}$$

Subcase 3: For m is odd & n is even

$$\begin{aligned}
\sigma(x_{2i-1,j}) &= 2j-1, \text{ for } 1 \le i \le \frac{m+1}{2}, 1 \le j \le 4;\\
\sigma(x_{2i,j}) &= 2j, \text{ for } 1 \le i \le \frac{m-1}{2}, 1 \le j \le 4;\\
\sigma(x_{2i-1,2j+1}) &= 3, \text{ for } 2 \le j \le \frac{n-2}{2}, 1 \le i \le \frac{m+1}{2};\\
\sigma(x_{2i-1,2j+2}) &= 5, \text{ for } 2 \le j \le \frac{n-2}{2}, 1 \le i \le \frac{m+1}{2};
\end{aligned}$$

$$\begin{aligned}
\sigma(x_{2i,2j+1}) &= 4, \text{ for } 2 \le j \le \frac{n-2}{2}, 1 \le i \le \frac{m-1}{2};\\
\sigma(x_{2i,2j+2}) &= 6, \text{ for } 2 \le j \le \frac{n-2}{2}, 1 \le i \le \frac{m-1}{2}.
\end{aligned}$$

Subcase 4: For m is even & n is odd

$$\begin{aligned}
\sigma(x_{2i-1,j}) &= 2j-1, \text{ for } 1 \le i \le \frac{m}{2}, 1 \le j \le 4;\\
\sigma(x_{2i,j}) &= 2j, \text{ for } 1 \le i \le \frac{m}{2}, 1 \le j \le 4;\\
\sigma(x_{2i-1,3j-1}) &= 3, \text{ for } 2 \le j \le \frac{n-3}{2}, 1 \le i \le \frac{m}{2};\\
\sigma(x_{2i-1,3j}) &= 5, \text{ for } 2 \le j \le \frac{n-3}{2}, 1 \le i \le \frac{m}{2};\\
\sigma(x_{2i-1,3j+1}) &= 7, \text{ for } 2 \le j \le \frac{n-3}{2}, 1 \le i \le \frac{m}{2};\\
\sigma(x_{2i,3j-1}) &= 4, \text{ for } 2 \le j \le \frac{n-3}{2}, 1 \le i \le \frac{m}{2};\\
\sigma(x_{2i,3j}) &= 6, \text{ for } 2 \le j \le \frac{n-3}{2}, 1 \le i \le \frac{m}{2};\\
\sigma(x_{2i,3j+1}) &= 8, \text{ for } 2 \le j \le \frac{n-3}{2}, 1 \le i \le \frac{m}{2}.
\end{aligned}$$

By assumption, $\chi_b(G[H]) \ge 8$. Let us assume that $\chi_b(G[H])$ is greater than 8. As wheel G increases the adjacency between any two vertex decreases (ie.,) $x_{2,4}, x_{2,5}, x_{3,5}, \ldots$ are not connected. So, the colors assigned to the corresponding vertices doesn't form any one of the color class which contradicts the definition of $b-$ coloring. So, $\chi_b(G[H]) \leq 8$. But, the $b-$chromatic number of $\chi_b(G[H])$ is the largest positive integer. Therefore $\chi_b(G[H]) = 8$, for $n \neq 5$.

Case 2: For $n = 5$

Subcase 1: For m is odd:

$$\begin{aligned}
\sigma(w_{2i-1,j}) &= 2j-1, \text{ for } 1 \leq i \leq \frac{m-1}{2}, 1 \leq j \leq 3;\\
\sigma(w_{2i,j}) &= 2j, \text{ for } 1 \leq i \leq \frac{m-1}{2}, 1 \leq j \leq 3;\\
\sigma(w_{2i-1,4}) &= 3, \text{ for } 1 \leq i \leq \frac{m-1}{2}; \sigma(w_{2i,4}) = 4, \text{ for } 1 \leq i \leq \frac{m-1}{2};\\
\sigma(w_{2i-1,5}) &= 5, \text{ for } 1 \leq i \leq \frac{m-1}{2}; \sigma(w_{2i,5}) = 6, \text{ for } 1 \leq i \leq \frac{m-1}{2}.
\end{aligned}$$

Subcase 2: For m is even:

$$\begin{aligned}
\sigma(w_{2i-1,j}) &= 2j-1, \text{ for } 1 \leq i \leq \frac{m}{2}, 1 \leq j \leq 3;\\
\sigma(w_{2i,j}) &= 2j, \text{ for } 1 \leq i \leq \frac{m}{2}, 1 \leq j \leq 3;\\
\sigma(w_{2i-1,4}) &= 3, \text{ for } 1 \leq i \leq \frac{m}{2}; \sigma(w_{2i,4}) = 4, \text{ for } 1 \leq i \leq \frac{m}{2};\\
\sigma(w_{2i-1,5}) &= 5, \text{ for } 1 \leq i \leq \frac{m}{2}; \sigma(w_{2i,5}) = 6, \text{ for } 1 \leq i \leq \frac{m}{2}.
\end{aligned}$$

By assumption, $\chi_b(G[H]) \geq 6$. Let us assume that $\chi_b(G[H])$ is greater than 6. As path P_m increases the adjacency between any 2 vertex decreases (ie.,) there is no connection between $x_{1,3}, x_{1,4}, x_{2,4}, \ldots\ldots$ and also in graph G. Hence, the colors assigned to the corresponding vertices doesn't form any one of the color class which contradicts the definition of $b-$ coloring. So, $\chi_b(G[H]) \leq 6$. But, the $b-$chromatic number of $\chi_b(G[H])$ is the largest positive integer. Therefore $\chi_b(G[H]) = 6$, for $n = 5$. □

We consider the graph G be the isomorphic to the cycle graph of order m vertices and H be the isomorphic to the wheel graph of order n vertices. Let $V(G) = \{u_i : 1 \leq i \leq m\}$ and $V(H) = \{v_j : 1 \leq j \leq 2n\}$, where $v_j, (j = 1, 2, \ldots n)$ are the vertices of cycle taken in a cyclic order and $v_{n+j}, (j = 1, 2, \ldots n)$ are pendant vertices such that each $v_j v_{n+j}$ are the pendant edge. Let $V(G[H]) = \bigcup_{i=1}^{m} \{x_{i,j} : 1 \leq j \leq 2n\}$, where $x_{i,j}$ are the vertices of $u_i v_j (1 \leq i \leq m, 1 \leq j \leq 2n)$.

Theorem 3.3. *The graph G be the isomorphic to the cycle graph of order m and H be the isomorphic to the wheel graph of order n. Then*

$$\chi_b(G[H]) = \begin{cases} 9, & for \quad n = 5 \;\&\; m \neq 4 \\ 8, & for \quad m = 4 \;\&\; n \neq 5 \\ 6, & for \quad m = 4 \;\&\; n = 5 \\ 12, & otherwise. \end{cases}$$

Proof. Define a mapping $\sigma : V(G[H]) \to N$ as follows:

Case 1: For $n = 5$ and $m \neq 4$.

$$\begin{aligned}
\sigma(x_{i,1}) &= i, \text{ for } 1 \leq i \leq 3; \sigma(x_{1,2j}) = 4, \text{ for } 1 \leq j \leq n-3;\\
\sigma(x_{2,2j}) &= 5, \text{ for } 1 \leq j \leq n-3; \sigma(x_{3,2j}) = 6, \text{ for } 1 \leq j \leq n-3;
\end{aligned}$$

$$
\begin{aligned}
\sigma(x_{1,2j+1}) &= 7,\ \text{for } 1 \le j \le n-3; \sigma(x_{2,2j+1}) = 8,\ \text{for } 1 \le j \le n-3;\\
\sigma(x_{3,2j+1}) &= 9,\ \text{for } 1 \le j \le n-3; \sigma(x_{m,1}) = 3;\\
\sigma(x_{2i,1}) &= 1,\ \text{for } 2 \le i \le \left\lceil \frac{m-2}{2} \right\rceil;\\
\sigma(x_{2i+1,1}) &= 2,\ \text{for } 2 \le i \le \left\lfloor \frac{m-2}{2} \right\rfloor;\\
\sigma(x_{2i,2j}) &= 4,\ \text{for } 2 \le i \le \left\lceil \frac{m-2}{2} \right\rceil,\ 1 \le j \le n-3;\\
\sigma(x_{2i+1,2j}) &= 5,\ \text{for } 2 \le i \le \left\lfloor \frac{m-2}{2} \right\rfloor,\ 1 \le j \le n-3;\\
\sigma(x_{2i,2j+1}) &= 7,\ \text{for } 2 \le i \le \left\lceil \frac{m-2}{2} \right\rceil,\ 1 \le j \le n-3;\\
\sigma(x_{2i+1,2j+1}) &= 8,\ \text{for } 2 \le i \le \left\lfloor \frac{m-2}{2} \right\rfloor,\ 1 \le j \le n-3;\\
\sigma(x_{m,2j}) &= 6,\ \text{for } 1 \le j \le n-3; \sigma(x_{m,2j+1}) = 9,\ \text{for } 1 \le j \le n-3.
\end{aligned}
$$

By assumption, $\chi_b(G[H]) \ge 9$. Let us assume that $\chi_b(G[H])$ is greater than 9. As $n = 5$ & m increases, the adjacency between any two vertex decreases. So, the colors assigned to the corresponding vertices doesn't form any one of the color class which contradicts the definition of $b-$ coloring. So, $\chi_b(G[H]) \le 9$. But, the $b-$chromatic number of $\chi_b(G[H])$ is the largest positive integer. Therefore $\chi_b(G[H]) = 9,$ for $n = 5$.

Case 2**:** For $m = 4$ & $n \ne 5$

$$
\begin{aligned}
\sigma(x_{2i-1,1}) &= 1,\ \text{for } 1 \le i \le m-2; \sigma(x_{2i,1}) = 2,\ \text{for } 1 \le i \le m-2;\\
\sigma(x_{2i-1,2}) &= 3,\ \text{for } 1 \le i \le m-2; \sigma(x_{2i,2}) = 4,\ \text{for } 1 \le i \le m-2;\\
\sigma(x_{2i-1,3}) &= 5,\ \text{for } 1 \le i \le m-2; \sigma(x_{2i,3}) = 6,\ \text{for } 1 \le i \le m-2;\\
\sigma(x_{2i-1,4}) &= 7,\ \text{for } 1 \le i \le m-2; \sigma(x_{2i,4}) = 8,\ \text{for } 1 \le i \le m-2;\\
\sigma(x_{2i-1,2j+3}) &= 3,\ \text{for } 1 \le i \le m-2\ \&\ 1 \le j \le \left\lfloor \frac{n-4}{2} \right\rfloor;\\
\sigma(x_{2i,2j+3}) &= 4,\ \text{for } 1 \le i \le m-2\ \&\ 1 \le j \le \left\lfloor \frac{n-4}{2} \right\rfloor;\\
\sigma(x_{2i-1,2j+4}) &= 5,\ \text{for } 1 \le i \le m-2\ \&\ 1 \le j \le \left\lceil \frac{n-6}{2} \right\rceil;\\
\sigma(x_{2i,2j+4}) &= 6,\ \text{for } 1 \le i \le m-2\ \&\ 1 \le j \le \left\lceil \frac{n-6}{2} \right\rceil;\\
\sigma(x_{2i-1,n}) &= 7,\ \text{for } 1 \le i \le m-2; \sigma(x_{2i,n}) = 8,\ \text{for } 1 \le i \le m-2.
\end{aligned}
$$

By assumption, $\chi_b(G[H]) \ge 8$. Let us assume that $\chi_b(G[H])$ is greater than 8. As $m = 4$ and W_n increases, the adjacency between any two vertex decreases. So, the colors assigned to the corresponding vertices doesn't form any one of the color class which contradicts the definition of $b-$ coloring. So, $\chi_b(G[H]) \le 8$. But, the $b-$chromatic number of $\chi_b(G[H])$ is the largest positive integer. Therefore $\chi_b(G[H]) = 8,$ for $m = 4$.

Case 3**:** For $m = 4$ and $n = 5$

$$
\begin{aligned}
\sigma(x_{2i-1,1}) &= 1,\ \text{for } 1 \le i \le m-2; \sigma(x_{2i,1}) = 2,\ \text{for } 1 \le i \le m-2;\\
\sigma(x_{2i-1,2j}) &= 3,\ \text{for } 1 \le i \le m-2, 1 \le j \le n-3;\\
\sigma(x_{2i,2j}) &= 4,\ \text{for } 1 \le i \le m-2, 1 \le j \le n-3;\\
\sigma(x_{2i-1,2j+1}) &= 5,\ \text{for } 1 \le i \le m-2, 1 \le j \le n-3;\\
\sigma(x_{2i,2j+1}) &= 6,\ \text{for } 1 \le i \le m-2, 1 \le j \le n-3.
\end{aligned}
$$

By assumption, $\chi_b(G[H]) \geq 6$. Let us assume that $\chi_b(G[H])$ is greater than 6. As $m = 4$ & $n = 5$ the edges between most of the two vertices are not connected. So, the colors assigned to the corresponding vertices doesn't form any one of the color class which contradicts the definition of $b-$ coloring. So, $\chi_b(G[H]) \leq 6$. But, the $b-$chromatic number of $\chi_b(G[H])$ is the largest positive integer. Therefore $\chi_b(G[H]) = 6$, for $m = 4, n = 5$.

Case 4: For $m \neq 4$ and $n \neq 5$

Subcase 1: For m is odd.

$$\begin{aligned}
\sigma(x_{i,1}) &= i, \text{ for } 1 \leq i \leq 3; \sigma(x_{i,2}) = i+3, \text{ for } 1 \leq i \leq 3;\\
\sigma(x_{i,3}) &= i+6, \text{ for } 1 \leq i \leq 3; \sigma(x_{i,4}) = i+9, \text{ for } 1 \leq i \leq 3;\\
\sigma(x_{2i,1}) &= 1, \text{ for } 2 \leq i \leq \frac{m-1}{2}; \sigma(x_{2i+1,1}) = 3, \text{ for } 2 \leq i \leq \frac{m-1}{2};\\
\sigma(x_{2i,2}) &= 4, \text{ for } 2 \leq i \leq \frac{m-1}{2}; \sigma(x_{2i+1,2}) = 6, \text{ for } 2 \leq i \leq \frac{m-1}{2};\\
\sigma(x_{2i,3}) &= 7, \text{ for } 2 \leq i \leq \frac{m-1}{2}; \sigma(x_{2i+1,3}) = 9, \text{ for } 2 \leq i \leq \frac{m-1}{2};\\
\sigma(x_{2i,4}) &= 10, \text{ for } 2 \leq i \leq \frac{m-1}{2}; \sigma(x_{2i+1,4}) = 12, \text{ for } 2 \leq i \leq \frac{m-1}{2};\\
\sigma(x_{1,n}) &= 10; \sigma(x_{2,n}) = 11; \sigma(x_{3,n}) = 12;\\
\sigma(x_{1,2j+3}) &= 4, \text{ for } 1 \leq j \leq \left\lfloor \frac{n-4}{2} \right\rfloor; \sigma(x_{2,2j+3}) = 5, \text{ for } 1 \leq j \leq \left\lfloor \frac{n-4}{2} \right\rfloor;\\
\sigma(x_{3,2j+3}) &= 6, \text{ for } 1 \leq j \leq \left\lfloor \frac{n-4}{2} \right\rfloor; \sigma(x_{1,2j+4}) = 7, \text{ for } 1 \leq j \leq \left\lceil \frac{n-6}{2} \right\rceil;\\
\sigma(x_{2,2j+4}) &= 8, \text{ for } 1 \leq j \leq \left\lceil \frac{n-6}{2} \right\rceil; \sigma(x_{3,2j+4}) = 9, \text{ for } 1 \leq j \leq \left\lceil \frac{n-6}{2} \right\rceil;\\
\sigma(x_{2i,2j+3}) &= 4, \text{ for } 2 \leq i \leq \frac{m-1}{2}, 1 \leq j \leq \left\lfloor \frac{n-4}{2} \right\rfloor;\\
\sigma(x_{2i+1,2j+3}) &= 6, \text{ for } 2 \leq i \leq \frac{m-1}{2}, 1 \leq j \leq \left\lfloor \frac{n-4}{2} \right\rfloor;\\
\sigma(x_{2i,2j+4}) &= 7, \text{ for } 2 \leq i \leq \frac{m-1}{2}, 1 \leq j \leq \left\lceil \frac{n-6}{2} \right\rceil;\\
\sigma(x_{2i+1,2j+4}) &= 9, \text{ for } 2 \leq i \leq \frac{m-1}{2}, 1 \leq j \leq \left\lceil \frac{n-6}{2} \right\rceil;\\
\sigma(x_{2i,n}) &= 10, \text{ for } 2 \leq i \leq \frac{m-1}{2}; \sigma(x_{2i+1,n}) = 12, \text{ for } 2 \leq i \leq \frac{m-1}{2}.
\end{aligned}$$

Subcase 2: For m is even.

$$\begin{aligned}
\sigma(x_{i,1}) &= i, \text{ for } 1 \leq i \leq 3; \sigma(x_{i,2}) = i+3, \text{ for } 1 \leq i \leq 3;\\
\sigma(x_{i,3}) &= i+6, \text{ for } 1 \leq i \leq 3; \sigma(x_{i,4}) = i+9, \text{ for } 1 \leq i \leq 3;\\
\sigma(x_{1,2j+3}) &= 4, \text{ for } 1 \leq j \leq \left\lfloor \frac{n-4}{2} \right\rfloor; \sigma(x_{2,2j+3}) = 5, \text{ for } 1 \leq j \leq \left\lfloor \frac{n-4}{2} \right\rfloor;\\
\sigma(x_{3,2j+3}) &= 6, \text{ for } 1 \leq j \leq \left\lfloor \frac{n-4}{2} \right\rfloor; \sigma(x_{1,2j+4}) = 7, \text{ for } 1 \leq j \leq \left\lceil \frac{n-6}{2} \right\rceil;\\
\sigma(x_{2,2j+4}) &= 8, \text{ for } 1 \leq j \leq \left\lceil \frac{n-6}{2} \right\rceil; \sigma(x_{3,2j+4}) = 9, \text{ for } 1 \leq j \leq \left\lceil \frac{n-6}{2} \right\rceil;
\end{aligned}$$

$$\begin{aligned}
\sigma(x_{1,n}) &= 10; \sigma(x_{2,n}) = 11; \sigma(x_{3,n}) = 12;\\
\sigma(x_{2i,1}) &= 1,\ \text{for } 2 \leq i \leq \frac{m-2}{2}; \sigma(x_{2i+1,1}) = 2,\ \text{for } 2 \leq i \leq \frac{m-2}{2};\\
\sigma(x_{2i,2}) &= 4,\ \text{for } 2 \leq i \leq \frac{m-2}{2}; \sigma(x_{2i+1,2}) = 5,\ \text{for } 2 \leq i \leq \frac{m-2}{2};\\
\sigma(x_{2i,3}) &= 7,\ \text{for } 2 \leq i \leq \frac{m-2}{2}; \sigma(x_{2i+1,3}) = 8,\ \text{for } 2 \leq i \leq \frac{m-2}{2};\\
\sigma(x_{2i,4}) &= 10,\ \text{for } 2 \leq i \leq \frac{m-2}{2}; \sigma(x_{2i+1,4}) = 11,\ \text{for } 2 \leq i \leq \frac{m-2}{2};\\
\sigma(x_{m,1}) &= 3; \sigma(x_{m,2}) = 6; \sigma(x_{m,3}) = 9; \sigma(x_{m,4}) = 12;\\
\sigma(x_{2i,2j+3}) &= 4,\ \text{for } 2 \leq i \leq \frac{m-2}{2}, 1 \leq j \leq \left\lfloor \frac{n-4}{2} \right\rfloor;\\
\sigma(x_{2i+1,2j+3}) &= 5,\ \text{for } 2 \leq i \leq \frac{m-2}{2}, 1 \leq j \leq \left\lfloor \frac{n-4}{2} \right\rfloor;\\
\sigma(x_{2i,2j+4}) &= 7,\ \text{for } 2 \leq i \leq \frac{m-2}{2}, 1 \leq j \leq \left\lceil \frac{n-6}{2} \right\rceil;\\
\sigma(x_{2i+1,2j+4}) &= 8,\ \text{for } 2 \leq i \leq \frac{m-2}{2}, 1 \leq j \leq \left\lceil \frac{n-6}{2} \right\rceil;\\
\sigma(x_{m,2j+3}) &= 6,\ \text{for } 1 \leq j \leq \left\lfloor \frac{n-4}{2} \right\rfloor; \sigma(x_{m,2j+4}) = 9,\ \text{for } 1 \leq j \leq \left\lceil \frac{n-6}{2} \right\rceil;\\
\sigma(x_{2i,n}) &= 10,\ \text{for } 2 \leq i \leq \frac{m-2}{2}; \sigma(x_{2i+1,n}) = 11,\ \text{for } 2 \leq i \leq \frac{m-2}{2};\\
\sigma(x_{m,n}) &= 12.
\end{aligned}$$

By assumption, $\chi_b(G[H]) \geq 12$. Let us assume that $\chi_b(G[H])$ is greater than 12. As C_m and W_n increases, the adjacency between any two vertex decreases. So, the colors assigned to the corresponding vertices doesn't form any one of the color class which contradicts the definitions of $b-$ coloring. So, $\chi_b(G[H]) \leq 12$. But, the $b-$chromatic number of $\chi_b(G[H])$ is the largest positive integer. Therefore $\chi_b(G[H]) = 12$. □

We consider the graph G and H be the isomorphic to the wheel graphs of order m and n vertices. Let $V(G) == \{u_1\} \cup \{u_i : 2 \leq i \leq m\}$ and $V(H) = \{v_1\} \cup \{v_j : 2 \leq j \leq n\}$. By the definition of lexicographic product, let $V(G[H]) = \bigcup_{i=1}^{m} \{x_{i,j} : 1 \leq j \leq n\}$; where $x_{i,j}$ are the vertices of $u_i v_j$ $(1 \leq i \leq m, 1 \leq j \leq n)$.

Theorem 3.4. *The graph G and H be the isomorphic to the wheel graphs of order m and n vertices. Then*

$$\chi_b(G[H]) = \begin{cases} 9, & for \quad m = n = 5 \\ 12, & for \quad m = 5 \ \&\ n \neq 5,\ n = 5 \ \&\ m \neq 5 \\ 16, & otherwise. \end{cases}$$

Proof. Define a mapping $\sigma : V(G[H]) \to N$ as follows:

Case 1: For $m = n = 5$

$$\begin{aligned}
\sigma(x_{1,1}) &= 1; \sigma(x_{2i,1}) = 2,\ \text{for } 1 \leq i \leq \frac{m-1}{2};\\
\sigma(x_{2i+1,1}) &= 3,\ \text{for } 1 \leq i \leq \frac{m-1}{2}; \sigma(x_{1,2j}) = 4,\ \text{for } 1 \leq j \leq \frac{n-1}{2};
\end{aligned}$$

$$\begin{aligned}
\sigma(x_{1,2j+1}) &= 7,\ \text{for } 1 \le j \le \frac{n-1}{2};\\
\sigma(x_{2i,2j}) &= 5,\ \text{for } 1 \le i \le \frac{m-1}{2}, 1 \le j \le \frac{n-1}{2};\\
\sigma(x_{2i,2j+1}) &= 8,\ \text{for } 1 \le i \le \frac{m-1}{2}, 1 \le j \le \frac{n-1}{2};\\
\sigma(x_{2i+1,2j}) &= 6,\ \text{for } 1 \le i \le \frac{m-1}{2}, 1 \le j \le \frac{n-1}{2};\\
\sigma(x_{2i,2j+1}) &= 9,\ \text{for } 1 \le i \le \frac{m-1}{2}, 1 \le j \le \frac{n-1}{2}.
\end{aligned}$$

By assumption, $\chi_b(G[H]) \ge 9$. Let us assume that $\chi_b(G[H])$ is greater than 9. As $m \& n = 5$, there is no adjacency between $x_{2,4}$ and $x_{3,5}$. So, the colors assigned to the corresponding vertices doesn't form any one of the color class which contradicts the definition of $b-$coloring. So, $\chi_b(G[H]) \le 9$. But, the $b-$chromatic number of $\chi_b(G[H])$ is the largest positive integer. Therefore $\chi_b(G[H]) = 9$, for $m \& n = 5$.

Case 2: For $m = 5$ & $n \ne 5$, $n = 5$ & $m \ne 5$

Subcase 1: For $n = 5$

$$\begin{aligned}
\sigma(x_{i,1}) &= i,\ \text{for } 1 \le i \le 4; \sigma(x_{1,2j}) = 5,\ \text{for } 1 \le j \le n-3;\\
\sigma(x_{1,2j+1}) &= 9,\ \text{for } 1 \le j \le n-3; \sigma(x_{2,2j}) = 6,\ \text{for } 1 \le j \le n-3;\\
\sigma(x_{2,2j+1}) &= 10,\ \text{for } 1 \le j \le n-3; \sigma(x_{3,2j}) = 7,\ \text{for } 1 \le j \le n-3;\\
\sigma(x_{3,2j+1}) &= 11,\ \text{for } 1 \le j \le n-3; \sigma(x_{4,2j}) = 8,\ \text{for } 1 \le j \le n-3;\\
\sigma(x_{4,2j+1}) &= 12,\ \text{for } 1 \le j \le n-3; \sigma(x_{m,1}) = 4;\\
\sigma(x_{m,2j}) &= 8,\ \text{for } 1 \le j \le n-3; \sigma(x_{m,2j+1}) = 12,\ \text{for } 1 \le j \le n-3;\\
\sigma(x_{2i+1,1}) &= 2,\ \text{for } 2 \le i \le \left\lfloor \frac{m-2}{2} \right\rfloor; \sigma(x_{2i+2,1}) = 3,\ \text{for } 2 \le i \le \left\lfloor \frac{m-4}{2} \right\rfloor;\\
\sigma(x_{2i+1,2j}) &= 6,\ \text{for } 2 \le i \le \left\lfloor \frac{m-2}{2} \right\rfloor, 1 \le j \le n-3;\\
\sigma(x_{2i+2,2j}) &= 7,\ \text{for } 2 \le i < \left\lfloor \frac{m-4}{2} \right\rfloor, 1 \le j \le n-3;\\
\sigma(x_{2i+1,2j+1}) &= 10,\ \text{for } 2 \le i \le \left\lfloor \frac{m-2}{2} \right\rfloor, 1 \le j \le n-3;\\
\sigma(x_{2i+2,2j+1}) &= 11,\ \text{for } 2 \le i < \left\lfloor \frac{m-4}{2} \right\rfloor, 1 \le j \le n-3.
\end{aligned}$$

Subcase 2: For $m = 5$

$$\begin{aligned}
\sigma(x_{1,j}) &= 3j-2 \text{ for } 1 \le j \le 4; \sigma(x_{2i,1}) = 2,\ \text{for } 1 \le i \le m-3;\\
\sigma(x_{2i+1,1}) &= 3,\ \text{for } 1 \le i \le m-3; \sigma(x_{2i,2}) = 5,\ \text{for } 1 \le i \le m-3;\\
\sigma(x_{2i+1,2}) &= 6,\ \text{for } 1 \le i \le m-3; \sigma(x_{2i,3}) = 8,\ \text{for } 1 \le i \le m-3;\\
\sigma(x_{2i+1,3}) &= 9,\ \text{for } 1 \le i \le m-3; \sigma(x_{2i,4}) = 11,\ \text{for } 1 \le i \le m-3;\\
\sigma(x_{2i+1,4}) &= 12,\ \text{for } 1 \le i \le m-3; \sigma(x_{1,n}) = 10;\\
\sigma(x_{2i,n}) &= 11,\ \text{for } 1 \le i \le m-3; \sigma(x_{2i+1,n}) = 12,\ \text{for } 1 \le i \le m-3;\\
\sigma(x_{1,2j+3}) &= 4,\ \text{for } 1 \le j \le \left\lceil \frac{n-3}{2} \right\rceil;\\
\sigma(x_{2i,2j+3}) &= 5,\ \text{for } 1 \le j \le \left\lceil \frac{n-3}{2} \right\rceil, 1 \le i \le m-3;
\end{aligned}$$

$$
\begin{aligned}
\sigma(x_{2i+1,2j+3}) &= 6, \text{ for } 1 \le j \le \left\lceil \frac{n-3}{2} \right\rceil, 1 \le i \le m-3; \\
\sigma(x_{1,2j+4}) &= 7, \text{ for } 1 \le j \le \left\lfloor \frac{n-5}{2} \right\rfloor; \\
\sigma(x_{2i,2j+4}) &= 8, \text{ for } 1 \le j \le \left\lfloor \frac{n-5}{2} \right\rfloor, 1 \le i \le m-3; \\
\sigma(x_{2i+1,2j+4}) &= 9, \text{ for } 1 \le j \le \left\lfloor \frac{n-5}{2} \right\rfloor, 1 \le i \le m-3.
\end{aligned}
$$

By assumption, $\chi_b(G[H]) \geq 12$. Let us assume that $\chi_b(G[H])$ is greater than 12. As m increases the adjacency between any 2 vertex decreases. So, the colors assigned to the corresponding vertices doesn't form any one of the color class which contradicts the definition of $b-$ coloring. So, $\chi_b(G[H]) \leq 12$. But, the $b-$chromatic number of $\chi_b(G[H])$ is the largest positive integer. Therefore $\chi_b(G[H]) = 12$.

Case 3: For $m \neq 5 \neq n$

$$
\begin{aligned}
\sigma(x_{1,j}) &= 4j-3, \text{ for } 1 \le j \le 4; \sigma(x_{2,j}) = 4j-2, \text{ for } 1 \le j \le 4; \\
\sigma(x_{3,j}) &= 4j-1, \text{ for } 1 \le j \le 4; \sigma(x_{4,j}) = 4j, \text{ for } 1 \le j \le 4; \\
\sigma(x_{i,n}) &= i+12, \text{ for } 1 \le i \le 4; \sigma(x_{m,j}) = 4j, \text{ for } 1 \le j \le 4; \\
\sigma(x_{1,2j+1}) &= 5, \text{ for } 2 \le j \le \left\lceil \frac{n-3}{2} \right\rceil; \\
\sigma(x_{1,2j+2}) &= 9, \text{ for } 2 \le j \le \left\lfloor \frac{n-3}{2} \right\rfloor; \\
\sigma(x_{2,2j+1}) &= 6, \text{ for } 2 \le j \le \left\lceil \frac{n-3}{2} \right\rceil; \\
\sigma(x_{2,2j+2}) &= 10, \text{ for } 2 \le j \le \left\lfloor \frac{n-3}{2} \right\rfloor; \\
\sigma(x_{3,2j+1}) &= 7, \text{ for } 2 \le j \le \left\lceil \frac{n-3}{2} \right\rceil; \\
\sigma(x_{3,2j+2}) &= 11, \text{ for } 2 \le j \le \left\lfloor \frac{n-3}{2} \right\rfloor; \\
\sigma(x_{4,2j+1}) &= 8, \text{ for } 2 \le j \le \left\lceil \frac{n-3}{2} \right\rceil; \\
\sigma(x_{4,2j+2}) &= 12, \text{ for } 2 \le j \le \left\lfloor \frac{n-3}{2} \right\rfloor; \\
\sigma(x_{2i+1,1}) &= 2, \text{ for } 2 \le i \le \left\lceil \frac{m-3}{2} \right\rceil; \\
\sigma(x_{2i+2,1}) &= 3, \text{ for } 2 \le i \le \left\lfloor \frac{m-3}{2} \right\rfloor; \\
\sigma(x_{2i+1,2}) &= 6, \text{ for } 2 \le i \le \left\lceil \frac{m-3}{2} \right\rceil; \\
\sigma(x_{2i+2,2}) &= 7, \text{ for } 2 \le i \le \left\lfloor \frac{m-3}{2} \right\rfloor; \\
\sigma(x_{2i+1,3}) &= 10, \text{ for } 2 \le i \le \left\lceil \frac{m-3}{2} \right\rceil; \\
\sigma(x_{2i+2,3}) &= 11, \text{ for } 2 \le i \le \left\lfloor \frac{m-3}{2} \right\rfloor;
\end{aligned}
$$

$$\begin{aligned}
\sigma(x_{2i+1,4}) &= 14,\ \text{for } 2 \le i \le \left\lceil \frac{m-3}{2} \right\rceil; \\
\sigma(x_{2i+2,4}) &= 15,\ \text{for } 2 \le i \le \left\lfloor \frac{m-3}{2} \right\rfloor; \\
\sigma(x_{2i+1,2j+1}) &= 6,\ \text{for } 2 \le i \le \left\lceil \frac{m-3}{2} \right\rceil, 2 \le j \le \left\lceil \frac{n-3}{2} \right\rceil; \\
\sigma(x_{2i+2,2j+1}) &= 7,\ \text{for } 2 \le i \le \left\lfloor \frac{m-3}{2} \right\rfloor, 2 \le j \le \left\lfloor \frac{n-3}{2} \right\rfloor; \\
\sigma(x_{2i+1,2j+2}) &= 10,\ \text{for } 2 \le i \le \left\lceil \frac{m-3}{2} \right\rceil, 2 \le j \le \left\lceil \frac{n-3}{2} \right\rceil; \\
\sigma(x_{2i+2,2j+2}) &= 11,\ \text{for } 2 \le i \le \left\lfloor \frac{m-3}{2} \right\rfloor, 2 \le j \le \left\lfloor \frac{n-3}{2} \right\rfloor; \\
\sigma(x_{2i+1,n}) &= 14,\ \text{for } 2 \le i \le \left\lceil \frac{m-3}{2} \right\rceil; \\
\sigma(x_{2i+2,n}) &= 15,\ \text{for } 2 \le i \le \left\lfloor \frac{m-3}{2} \right\rfloor; \\
\sigma(x_{m,2j+1}) &= 8,\ \text{for } 2 \le j \le \left\lceil \frac{n-3}{2} \right\rceil; \\
\sigma(x_{m,2j+2}) &= 12,\ \text{for } 2 \le j \le \left\lfloor \frac{n-3}{2} \right\rfloor; \\
\sigma(x_{m,n}) &= 16.
\end{aligned}$$

By assumption, $\chi_b(G[H]) \ge 16$. Let us assume that $\chi_b(G[H])$ is greater than 16. As wheel W_n increases the adjacency between any 2 vertex decreases (ie.,) $x_{2,4}, x_{2,5}, x_{3,5}, \ldots$ are not connected. So, the colors assigned to the corresponding vertices doesn't form any one of the color class which contradicts the definition of $b-$ coloring. So, $\chi_b(G[H]) \le 16$. But, the $b-$chromatic number of $\chi_b(G[H])$ is the largest positive integer. Therefore $\chi_b(G[H]) = 16$, for $m \& n \ne 5$. □

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