Palestine Journal of Mathematics

Vol. 10 (Special Issue II, 2021) , 122–130

Rainbow dominator coloring in graphs

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Communicated by N. Mohanapriya

MSC 2010 Classifications: Primary 20M99, 13F10; Secondary 13A15, 13M05.

Keywords and phrases: rainbow coloring, dominator coloring, rainbow dominator coloring.

Abstract Let G be a connected graph. The rainbow vertex connection number of G is the minimum number of colors necessary to color G such that for each pair of its vertices, there is a path connecting them whose internal vertices are assigned distinct colors. A dominator coloring of G is a proper coloring in which each vertex of G dominates every vertex of some color class. The dominator chromatic number is the minimum number of color classes in a dominator coloring. In this paper, we introduce a new vertex coloring parameter called rainbow dominator chromatic number and determine this parameter for some standard graphs.

1 Introduction

In graph theory there exists two coloring problems. One is a vertex coloring problem and the other is an edge coloring problem. In both these types, various coloring parameters have been introduced and studied in detail and results related to these parameters are available in literature. A relatively new concept in edge coloring called rainbow coloring was introduced in the year 2008 by Chatrand, Johns and Mckeon [2] . An analogous vertex coloring concept called rainbow vertex coloring was introduced by Krivelevich and Yuster in the year 2010 [3].

All the graphs considered in this paper are connected, finite and undirected graphs. We refer [1] for all standard notations and terminologies.

In this paper, we introduce a new vertex coloring parameter called rainbow dominator chromatic number and determine this parameter for some standard graphs. We will denote rainbow dominator coloring by RDC throughout this paper.

Definition 1. A dominator coloring [4] of G is a proper coloring in which each vertex of G dominates every vertex of some color class. The dominator chromatic number $\chi_d(G)$ is the minimum number of color classes in a dominator coloring of a graph G.

In figure 1, a graph G with dominator chromatic number 3 is shown. In this figure, C_1, C_2, C_3 are the color classes.

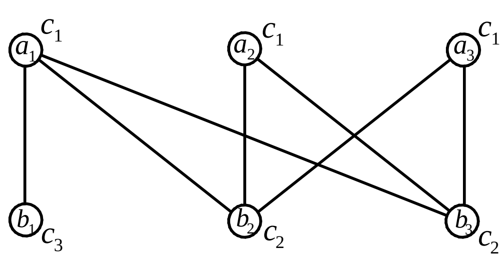

Figure 1. A graph G with $\chi_d(G) = 3$

Definition 2. A rainbow dominator coloring of a graph G is a proper rainbow coloring of G in which each vertex of G dominates every vertex of some color class. The rainbow dominator chromatic number, denoted by $\chi_{rd}(G)$ is the minimum number of color classes in a rainbow dominator coloring of G.

In figure 2, a RDC with ten color classes C_1, C_2, C_3,....C_{10} of the prism graph Y_{10} is shown.

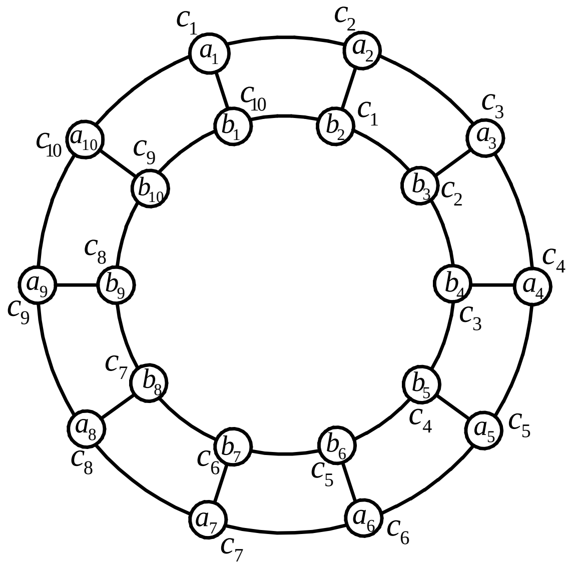

Figure 2. $\chi_{rd}(Y_{10}) = 10$

In this section, we determine $\chi_{rd}(G)$ for some standard graphs like the path P_n, the cycle C_n, the wheel $W_{1,n}$ and the prism graph Y_n.

We begin with the following result.

Theorem 1.1. *For* $n \geq 3$, $\chi_{rd}(P_n) = n - 1$.

Proof. Let $G = P_n$, $V(G) = \{a_x : 1 \leq x \leq n\}$ and $E(G) = \{e_x : e_x = (a_x, a_{x+1}), 1 \leq x \leq n-1\}$.
We allocate a RDC to the vertices of G as follows.

Consider the set $C : \{c_1, c_2, \cdots, c_{n-1}\}$ of color classes as follows:

$$C = \begin{cases} c_1 & \text{for} \quad a_x; x = 1, n \\ c_x & \text{for} \quad a_x; 2 \leq x \leq n-1 \end{cases}$$

Here, for $x = 1, n$, the vertices a_x of the color class c_1 dominate the vertices a_x of the color class c_x for $x = 2, n-1$. Also, for $2 \leq x \leq n-1$, the vertices a_x of the color class c_x dominate themselves.

Thus, $\chi_{rd}(P_n) \leq n - 1$.

To prove that $\chi_{rd}(P_n) \geq n-1$, we assume that $\chi_{rd}(P_n) = n-2$. Then $n-2$ colors must be allocated to the vertices of G for a RDC.

Here G is a path P_n, and it requires $n-1$ colors for a proper rainbow coloring.

A simple verification shows that there exist some paths (like the path: $a_{n-3} - a_{n-2} - a_{n-1} - a_n$) in which two of its vertices are colored with the same colors.

This is a contradiction.

Therefore, $\chi_{rd}(P_n) \geq n - 1$.

Thus, $\chi_{rd}(P_n) = n - 1$.

□

In our next result, we determine the rainbow dominator chromatic number of the cycle C_n.

Theorem 1.2. *For $k \geq 1$,*

$$\chi_{rd}(C_n) = \begin{cases} 7 & \textit{for } n = 9,10 \\ 8 & \textit{for } n = 11,12 \\ 4k+5 & \textit{for } n = 6k+7 \\ 4k+6 & \textit{for } n = 6k+8 \\ 4k+7 & \textit{for } n = 6k+9 \\ 4k+8 & \textit{for } n = 6k+10,\ 6k+11,\ 6k+12 \end{cases}$$

Proof. Let $G = C_n$. Let $V(G) = \{a_1, a_2, a_3, \cdots a_n, a_{n+1} = a_1\}$ and $E(G) = \{e_x : e_x = (a_x, a_{x+1})$ for $1 \leq x \leq n\}$.

We allocate a RDC to the vertices of G as follows.

- For $n = 4$ an *RDC* with color classes c_1 for $\{a_1, a_3\}$ and c_2 for $\{a_2, a_4\}$ gives $\chi_{rd}(C_4) = 2$.
- For $n = 5$ an *RDC* with color classes c_1 for $\{a_1, a_4\}$, c_2 for $\{a_2, a_5\}$ and c_3 for $\{a_3\}$ gives $\chi_{rd}(C_5) = 3$.
- For $n = 6$ an *RDC* with color classes c_1 for $\{a_1, a_4\}$, c_2 for $\{a_2\}$, c_3 for $\{a_3, a_6\}$ and c_4 for $\{a_5\}$ gives $\chi_{rd}(C_6) = 4$.
- For $n = 7$ an *RDC* with color classes c_1 for $\{a_1, a_4\}$, c_2 for $\{a_2\}$ and c_3 for $\{a_3, a_6\}$ c_4 for $\{a_5\}$ and c_5 for $\{a_7\}$ gives $\chi_{rd}(C_7) = 5$.
- For $n = 8$ an *RDC* with color classes c_1 for $\{a_1, a_5\}$, c_2 for $\{a_2\}$ and c_3 for $\{a_3, a_7\}$, c_4 for $\{a_6\}$, c_5 for $\{a_4\}$, c_6 for $\{a_8\}$ gives $\chi_{rd}(C_8) = 6$.

Case 1: $n = 9, 10$.

Consider the set $C : \{c_1, c_2, \cdots, c_7\}$ of color classes as follows:

$$C = \begin{cases} c_{\lceil \frac{n}{2} \rceil} & \text{for } a_{\lceil \frac{n}{3} \rceil + 1} \text{ for } n = 9 \\ c_{\lceil \frac{n}{2} \rceil} & \text{for } a_{\lceil \frac{n}{2} \rceil}, a_{n-1} \text{ for } n = 10 \\ c_{x+1} & \text{for } a_{2x+1}, a_{2x+6}, \text{ for } 0 \leq x \leq 1 \\ c_{x+3} & \text{for } a_{5x+2} \text{ for } 0 \leq x \leq \ 1 \\ c_{x+5} & \text{for } a_{x+4} \text{ for } x = 1 \\ c_{\lceil \frac{n}{2} \rceil + 2} & \text{for } a_n \end{cases}$$

Here, for each $0 \leq x \leq 1$, the vertices a_{2x+1}, a_{2x+6} of the color class c_{x+1} dominate the vertices a_{5x+2} of the color class c_{x+3}. Further for $n = 10$ the vertices $a_{\lceil \frac{n}{2} \rceil}, a_{n-1}$ of the color class $c_{\lceil \frac{n}{2} \rceil}$ dominate the vertex a_{x+4} of the color class c_{x+5} $for\ x = 1$ and the vertex a_n of the color class $c_{\lceil \frac{n}{2} \rceil + 2}$. Also, the vertices of the remaining color classes dominate themselves.

Thus, $\chi_{rd}(C_n) \leq 7$.

To prove that $\chi_{rd}(C_n) \geq 7$, we assume that $\chi_{rd}(C_n) = 6$. Then 6 colors must be allocated to the vertices of G for a RDC.

Here G is a cycle C_9 or C_{10}, which requires 5 distinct colors for a rainbow coloring. Even if we allocate 6 colors to the vertices of G, a simple verification shows that there exist some vertices in the graph G (For example, in C_9 the vertices $a_5\ and\ a_9$) which will not dominate every vertex of any color class.

This is a contradiction.

Therefore, $\chi_{rd}(C_n) \geq 7$.

Thus, $\chi_{rd}(C_n) = 7$.

Case 2: $n = 11, 12$.

Consider the set $C : \{c_1, c_2, \cdots, c_8\}$ of color classes as follows:

$$C = \begin{cases} c_{\lceil \frac{n}{2} \rceil} & \text{for } a_{\lceil \frac{n}{2} \rceil} \text{ for } n = 11 \\ c_{\lceil \frac{n}{2} \rceil} & \text{for } a_{\lceil \frac{n}{2} \rceil}, a_n \text{ for } n = 12 \\ c_{4x+1} & \text{for } a_{3x+1}, a_{\lfloor \frac{n+3}{2} \rfloor + 3x} \text{ for } 0 \le x \le 1 \\ c_2 & \text{for } a_3, a_9 \\ c_{4x-1} & \text{for } a_{3x-1} \text{ for } 1 \le x \le 2 \\ c_{4x+4} & \text{for } a_{\lfloor \frac{n+5}{2} \rfloor + 3x} \text{ for } 0 \le x \le 1 \end{cases}$$

Here, the vertices of the color class c_2 and for each $0 \le x \le 1$ and the vertices of the color class c_{4x+1} dominate the vertices of the color class c_{4x-1} and c_{4x+4} for $1 \le x \le 2$ and $0 \le x \le 1$ respectively . Further for $n = 12$ the vertices of the color class $c_{\lceil \frac{n}{2} \rceil}$ dominate the vertices of the color class c_{4x-1} and c_{4x+4} for $x = 2$ and $x = 1$ respectively. Also, the vertices of the remaining color classes dominate themselves.

Thus, $\chi_{rd}(C_n) \le 8$.

To prove that $\chi_{rd}(C_n) \ge 8$, we assume that $\chi_{rd}(C_n) = 7$. Then 7 colors must be allocated to the vertices of G for a RDC.

Here G is a cycle C_{11} or C_{12}, which requires 6 distinct colors for a rainbow coloring. Even if we allocate 7 colors to the vertices of G, a simple verification shows that there exist some vertices in the graph G (For example, in C_{12} the vertices a_5 and a_{11}) which will not dominate every vertex of any color class.

This is a contradiction.

Therefore, $\chi_{rd}(C_n) \ge 8$.

Thus, $\chi_{rd}(C_n) = 8$.

Case 3: $n = 6k + 7$.

Consider the set C of color classes $C : \{c_1, c_2, \cdots, c_{4k+5}\}$ as follows:

$$C = \begin{cases} c_{4x+1} & \text{for } a_{3x+1}, a_{\frac{n+1}{2} + 3x} \text{ for } 0 \le x \le \frac{n-7}{6} \\ c_{4x+2} & \text{for } a_{3x+3}, a_{\frac{n+5}{2} + 3x} \text{ for } 0 \le x \le \frac{n-7}{6} \\ c_{4x-1} & \text{for } a_{3x-1} \text{ for } 1 \le x \le \frac{n-1}{6} \\ c_{4x+4} & \text{for } a_{\frac{n+3}{2} + 3x} \text{ for } 0 \le x \le\ 1 \\ c_{4x+12} & \text{for } a_{\frac{n+15}{2} + 3x} \text{ for } 0 \le x \le \frac{n-19}{6}, x \ge 1 \\ c_{\frac{2n+1}{3}} & \text{for } a_n \end{cases}$$

Here, for $0 \le x \le \frac{n-7}{6}$, the vertices $a_{3x+1}, a_{\frac{n+1}{2}+3x}$ of the color class c_{4x+1} and the vertices $a_{3x+3}, a_{\frac{n+5}{2}+3x}$ of the color class c_{4x+2} dominate the vertices a_{3x-1} of the color class c_{4x-1} for $1 \le x \le \frac{n-1}{6}$ or the vertices $a_{\frac{n+3}{2}+3x}$ of the color class c_{4x+4} for $0 \le x \le\ 1$ or the vertices $a_{\frac{n+15}{2}+3x}$ of the color class c_{4x+12} for $0 \le x \le \frac{n-19}{6}$. Also, the vertices of the remaining color classes dominate themselves.

Thus, $\chi_{rd}(C_n) \le 4k + 5$.

To prove that $\chi_{rd}(C_n) \ge 4k + 5$, we assume that $\chi_{rd}(C_n) = 4k + 4$. Then $4k + 4$ colors must be allocated to the vertices of G for a RDC.

As G is a cycle of length n, it requires $\lceil \frac{n}{2} \rceil$ colors for a rainbow coloring. Even if we allocate $4k + 4$ colors to the vertices of G, a simple verification shows that there exist some vertices in the graph G (For example, in C_{13} the vertices a_{11} and a_{13}) which will not dominate every vertex of any color class.

This is a contradiction.

Therefore, $\chi_{rd}(C_n) \ge 4k + 5$.

Thus, $\chi_{rd}(C_n) = 4k + 5$.

Case 4: $n = 6k + 8$.

Consider the set $C : \{c_1, c_2, \cdots, c_{4k+6}\}$ of color classes as follows:

$$C = \begin{cases} c_{4x+1} & \text{for } a_{3x+1}, a_{\frac{n+2}{2}+3x} \text{ for } 0 \leq x \leq \frac{n-8}{6} \\ c_{4x+2} & \text{for } a_{3x+3}, a_{\frac{n+6}{2}+3x} \text{ for } 0 \leq x \leq \frac{n-8}{6} \\ c_{4x-1} & \text{for } a_{3x-1} \text{ for } 1 \leq x \leq \frac{n-2}{6} \\ c_{4x+4} & \text{for } a_{\frac{n+4}{2}+3x} \text{ for } 0 \leq x \leq \frac{n-8}{6} \\ c_{4x+5} & \text{for } a_{3x+4} \text{ for } x = \frac{n-8}{6} \\ c_{\frac{2n+2}{3}} & \text{for } a_n \end{cases}$$

Here, for $0 \leq x \leq \frac{n-8}{6}$, the vertices $a_{3x+1}, a_{\frac{n+2}{2}+3x}$ of the color class c_{4x+1} and the vertices $a_{3x+3}, a_{\frac{n+6}{2}+3x}$ of the color class c_{4x+2} dominate the vertices a_{3x-1} of the color class c_{4x-1} for $1 \leq x \leq \frac{n-2}{6}$ or the vertices $a_{\frac{n+4}{2}+3x}$ of the color class c_{4x+4} for $0 \leq x \leq \frac{n-8}{6}$ or the vertices a_{3x+4} of the color class c_{4x+5} for $x = \frac{n-8}{6}$. Also, the vertices of the remaining color classes dominate themselves.

Thus, $\chi_{rd}(C_n) \leq 4k + 6$.

To prove that $\chi_{rd}(C_n) \geq 4k + 6$, we assume that $\chi_{rd}(C_n) = 4k + 5$. Then $4k + 5$ colors must be allocated to the vertices of G for a RDC.

As G is a cycle of length n, it requires $\lceil \frac{n}{2} \rceil$ colors for a rainbow coloring. Even if we allocate $4k + 5$ colors to the vertices of G, a simple verification shows that there exist some vertices in the graph G, which will not dominate every vertex of any color class.

This is a contradiction.

Therefore, $\chi_{rd}(C_n) \geq 4k + 6$.

Thus, $\chi_{rd}(C_n) = 4k + 6$.

Case 5: $n = 6k + 9$.

Consider the set $C : \{c_1, c_2, \cdots, c_{4k+7}\}$ of color classes as follows:

$$C = \begin{cases} c_{4x+1} & \text{for } a_{3x+1}, a_{\frac{n+1}{2}+3x} \text{ for } 0 \leq x \leq \frac{n-9}{6} \\ c_{4x+2} & \text{for } a_{3x+3}, a_{\frac{n+5}{2}+3x} \text{ for } 0 \leq x \leq \frac{n-9}{6} \\ c_{4x-1} & \text{for } a_{3x-1} \text{ for } 1 \leq x \leq \frac{n-3}{6} \\ c_{4x+4} & \text{for } a_{\frac{n+3}{2}+3x} \text{ for } 0 \leq x \leq \frac{n-9}{6} \\ c_{4x+5} & \text{for } a_{3x+4} \text{ for } x = \frac{n-9}{6} \\ c_{4x+6} & \text{for } a_{6x+8} \text{ for } x = \frac{n-9}{6} \\ c_{\frac{2n+3}{3}} & \text{for } a_n \end{cases}$$

Here, for $0 \leq x \leq \frac{n-9}{6}$, the vertices $a_{3x+1}, a_{\frac{n+1}{2}+3x}$ of the color class c_{4x+1} and the vertices $a_{3x+3}, a_{\frac{n+5}{2}+3x}$ of the color class c_{4x+2} dominate the vertices a_{3x-1} of the color class c_{4x-1} for $1 \leq x \leq \frac{n-3}{6}$ or the vertices $a_{\frac{n+3}{2}+3x}$ of the color class c_{4x+4} for $0 \leq x \leq \frac{n-9}{6}$ or the vertices a_{3x+4} of the color class c_{4x+5} for $x = \frac{n-9}{6}$ or the vertices a_{6x+8} of the color class c_{4x+6} for $x = \frac{n-9}{6}$. Also, the vertices of the remaining color classes dominate themselves.

Thus, $\chi_{rd}(C_n) \leq 4k + 7$.

To prove that $\chi_{rd}(C_n) \geq 4k + 7$, we assume that $\chi_{rd}(C_n) = 4k + 6$. Then $4k + 6$ colors must be allocated to the vertices of G for a RDC.

As G is a cycle of length n, it requires $\lceil \frac{n}{2} \rceil$ colors for a rainbow coloring. Even if we allocate $4k + 6$ colors to the vertices of G, a simple verification shows that there exist some vertices in the graph G, which will not dominate every vertex of any color class.

This is a contradiction.

Therefore, $\chi_{rd}(C_n) \geq 4k + 7$.

Thus, $\chi_{rd}(C_n) = 4k + 7$.

Case 6: $n = 6k + 10$.

Consider the set $C : \{c_1, c_2, \cdots, c_{4k+8}\}$ of color classes as follows:

$$C = \begin{cases} c_{4x+1} & \text{for } a_{3x+1}, a_{\frac{n+2}{2}+3x} \text{ for } 0 \le x \le \frac{n-10}{6} \\ c_{4x+2} & \text{for } a_{3x+3}, a_{\frac{n+6}{2}+3x} \text{ for } 0 \le x \le \frac{n-10}{6} \\ c_{4x-1} & \text{for } a_{3x-1} \text{ for } 1 \le x \le \frac{n+2}{6} \\ c_{4x+4} & \text{for } a_{\frac{n+4}{2}+3x} \text{ for } 0 \le x \le \frac{n-10}{6} \\ c_{4x+5} & \text{for } a_{3x+4} \text{ for } x = \frac{n-10}{6} \\ c_{4x+6} & \text{for } a_{6x+9} \text{ for } x = \frac{n-10}{6} \\ c_{\frac{2n+4}{3}} & \text{for } a_n \end{cases}$$

Here, for $0 \le x \le \frac{n-10}{6}$, the vertices $a_{3x+1}, a_{\frac{n+2}{2}+3x}$ of the color class c_{4x+1} and the vertices $a_{3x+3}, a_{\frac{n+6}{2}+3x}$ of the color class c_{4x+2} dominate the vertices a_{3x-1} of the color class c_{4x-1} for $1 \le x \le \frac{n+2}{6}$ or the vertices $a_{\frac{n+4}{2}+3x}$ of the color class c_{4x+4} for $0 \le x \le \frac{n-10}{6}$ or the vertices a_{3x+4} of the color class c_{4x+5} for $x = \frac{n-10}{6}$ or the vertices a_{6x+9} of the color class c_{4x+6} for $x = \frac{n-10}{6}$. Also, the vertices of the remaining color classes dominate themselves.

Thus, $\chi_{rd}(C_n) \le 4k+8$.

To prove that $\chi_{rd}(C_n) \ge 4k+8$, we assume that $\chi_{rd}(C_n) = 4k+7$. Then $4k+7$ colors must be allocate to the vertices of G for a RDC.

As G is a cycle of length n, it requires $\lceil \frac{n}{2} \rceil$ colors for a rainbow coloring. Even if we allocate $4k+7$ colors to the vertices of G, a simple verification shows that there exist some vertices in the graph G, which will not dominate every vertex of any color class.

This is a contradiction.

Therefore, $\chi_{rd}(C_n) \ge 4k+8$.

Thus, $\chi_{rd}(C_n) = 4k+8$.

Case 7: $n = 6k+11$.

Consider the set $C : \{c_1, c_2, \cdots, c_{4k+8}\}$ of color classes as follows:

$$C = \begin{cases} c_{4x+1} & \text{for } a_{3x+1}, a_{\frac{n+1}{2}+3x} \text{ for } 0 \le x \le \frac{n-11}{6} \\ c_{4x+2} & \text{for } a_{3x+3}, a_{\frac{n+5}{2}+3x} \text{ for } 0 \le x \le \frac{n-11}{6} \\ c_{4x-1} & \text{for } a_{3x-1} \text{ for } 1 \le x \le \frac{n-5}{6} \\ c_{4x+4} & \text{for } a_{\frac{n+3}{2}+3x} \text{ for } 0 \le x \le \frac{n-11}{6} \\ c_{4x+5} & \text{for } (a_{3x+4}, a_{n-2}) \text{ for } x = \frac{n-11}{6} \\ c_{4x+6} & \text{for } a_{5x+3} \text{ for } x = \frac{n-11}{6} \\ c_{4x+7} & \text{for } a_{3x+5} \text{ for } x = \frac{n-11}{6} \\ c_{\frac{2n+2}{3}} & \text{for } a_n \end{cases}$$

Here, for $0 \le x \le \frac{n-11}{6}$, the vertices $a_{3x+1}, a_{\frac{n+1}{2}+3x}$ of the color class c_{4x+1} , the vertices $a_{3x+3}, a_{\frac{n+5}{2}+3x}$ of the color class c_{4x+2} and for $x = \frac{n-11}{6}$ the vertices (a_{3x+4}, a_{n-2}) of the color class c_{4x+5} dominate the vertices a_{3x-1} of the color class c_{4x-1} for $0 \le x \le \frac{n-5}{6}$ or the vertices $a_{\frac{n+3}{2}+3x}$ of the color class c_{4x+4} for $x = \frac{n-11}{6}$ or the vertices a_{5x+3} of the color class c_{4x+6} for $x = \frac{n-11}{6}$ or the vertices a_{3x+5} of the color class c_{4x+7} for $x = \frac{n-11}{6}$. Also, the vertices of the remaining color classes dominate themselves.

Thus, $\chi_{rd}(C_n) \le 4k+8$.

To prove that $\chi_{rd}(C_n) \ge 4k+8$, we assume that $\chi_{rd}(C_n) = 4k+7$. Then $4k+7$ colors must be allocated to the vertices of G for a RDC.

As G is a cycle of length n, it requires $\lceil \frac{n}{2} \rceil$ colors for a rainbow coloring. Even if we allocate $4k+7$ colors to the vertices of G, a simple verification shows that there exist some vertices in the graph G, which will not dominate every vertex of any color class.

This is a contradiction.

Therefore, $\chi_{rd}(C_n) \ge 4k+8$.

Thus, $\chi_{rd}(C_n) = 4k + 8$.

Case 8: $n = 6k + 12$.
Consider the set $C : \{c_1, c_2, \cdots, c_{4k+8}\}$ of color classes as follows:

$$C = \begin{cases} c_{4x+1} & \text{for } a_{3x+1}, a_{\frac{n+2}{2}+3x} \text{ for } 0 \leq x \leq \frac{n-6}{6} \\ c_{4x+2} & \text{for } a_{3x+3}, a_{\frac{n+6}{2}+3x} \text{ for } 0 \leq x \leq \frac{n-6}{6} \\ c_{4x-1} & \text{for } a_{3x-1} \text{ for } 1 \leq x \leq \frac{n}{6} \\ c_{4x+4} & \text{for } a_{\frac{n+4}{2}+3x} \text{ for } 0 \leq x \leq \frac{n-6}{6} \end{cases}$$

Here, for $0 \leq x \leq \frac{n-6}{6}$, the vertices $a_{3x+1}, a_{\frac{n+2}{2}+3x}$ of the color class c_{4x+1} and the vertices $(a_{3x+3}, a_{\frac{n+6}{2}+3x})$ of the color class c_{4x+2} dominate the vertices a_{3x-1} of the color class c_{4x-1} for $1 \leq x \leq \frac{n}{6}$ or the vertices $a_{\frac{n+4}{2}+3x}$ of the color class c_{4x+4} for $0 \leq x \leq \frac{n-6}{6}$. Also, the vertices of the remaining color classes dominate themselves.

Thus, $\chi_{rd}(C_n) \leq 4k + 8$.

To prove that $\chi_{rd}(C_n) \geq 4k + 8$, we assume that $\chi_{rd}(C_n) = 4k + 7$. Then $4k + 7$ colors must be allocated to the vertices of G for a RDC.

As G is a cycle of length n, it requires $\lceil \frac{n}{2} \rceil$ colors for a rainbow coloring. Even if we allocate $4k + 7$ colors to the vertices of G, a simple verification shows that there exist some vertices in the graph G, which will not dominate every vertex of any color class.

This is a contradiction.

Therefore, $\chi_{rd}(C_n) \geq 4k + 8$.

Thus, $\chi_{rd}(C_n) = 4k + 8$.

□

In the next theorem, we determine the rainbow dominator chromatic number of the wheel graph $W_{1,n}$.

Theorem 1.3. *For $n \geq 3$,*

$$\chi_{rd}(W_{1,n}) = \begin{cases} 3 & \textit{if } n \textit{ is even} \\ 4 & \textit{if } n \textit{ is odd}. \end{cases}$$

Proof. Let $G = W_{1,n}$. Let G consists of an $n-$ cycle, $C_n = \{a_1, a_2, a_3, \cdots, a_n, a_{n+1} = v_1\}$ and another vertex b joined to every vertex of C_n .

let $E(G) = E_A \cup E_B$, where for $1 \leq x \leq n$

- $E_A = \{e_x : e_x = (a_x, a_{x+1})\}$
- $E_B = \{e'_x : e'_x = (b, a_x)\}$

We allocate a RDC to the vertices of G as follows.

Case 1: n is even .
Consider the set $C : \{c_1, c_2, c_3\}$ of color classes as follows

$$C = \begin{cases} c_1 & \text{for } a_{2x-1}, \text{for } 1 \leq x \leq \frac{n}{2} \\ c_2 & \text{for } a_{2x}, \text{for } 1 \leq x \leq \frac{n}{2} \\ c_3 & \text{for } b \end{cases}$$

Here, for each $1 \leq x \leq \frac{n}{2}$, the vertices a_{2x-1} of the color class c_1 and the vertices a_{2x} of the color class c_2 dominate the vertex b of the color classes c_3. Also, the vertex b of the color class c_3 dominates itself.

Thus, $\chi_{rd}(W_{1,n}) \leq 3$.

To prove that $\chi_{rd}(W_{1,n}) \geq 3$, we assume that $\chi_{rd}(W_{1,n}) = 2$. Then two colors must be allocated to the vertices of G for a RDC.

As G contains an $n-$ cycle (C_n), allocate these two colors to the vertices of C_n. Further, if we allocate any of these two colors to the vertex b, then a simple verification shows that there exist some paths (like the paths; $a_x - b - a_{x+1}, 1 \leq x \leq \frac{n}{2}$ in which two of its vertices are colored with the same colors.

This is a contradiction.

Therefore, $\chi_{rd}(W_{1,n}) \geq 3$.

Thus, $\chi_{rd}(W_{1,n}) = 3$.

Case 2: n is odd .

Consider the set $C : \{c_1, c_2, c_3, c_4\}$ of color classes as follows.

$$C = \begin{cases} c_1 & \text{for } a_{2x-1}, \text{ for } 1 \leq x \leq \lfloor \frac{n}{2} \rfloor \\ c_2 & \text{for } a_{2x}, \text{ for } 1 \leq x \leq \lfloor \frac{n}{2} \rfloor \\ c_3 & \text{for } a_{2n-1} \\ c_4 & \text{for } b \end{cases}$$

Here, for each $1 \leq x \leq \lfloor \frac{n}{2} \rfloor$, the vertices a_{2x-1} of the color class c_1 and the vertices a_{2x} of the color class c_2 dominate the vertex b of the color class c_4 and the vertex a_{2n-1} of the color class c_3 dominate the vertex b of the color class c_4. Also, the vertex b of the color class c_4 dominates itself.

Thus, $\chi_{rd}(W_{1,n}) \leq 4$. Now, to prove that $\chi_{rd}(W_{1,n}) \geq 4$, we assume that $\chi_{rd}(W_{1,n}) = 3$. Then three colors must be allocated to the vertices of G for a RDC.

As G contains an $n-$ cycle (C_n), assign these three colors to the vertices of C_n. Further, if we allocate any of these three colors to the vertex u, a simple verification shows that there exist some paths in which two of its vertices are colored with the same colors.

This is a contradiction.

Therefore, $\chi_{rd}(W_{1,n}) \geq 4$.

Thus, $\chi_{rd}(W_{1,n}) = 4$.

□

For the prism graph Y_n, we have the following result.

Theorem 1.4. *For* $n \geq 4$, $\chi_{rd}(Y_n) = n$.

Proof. Let $G = Y_n$. $V(G) = V_1 \cup V_2$ where

- $V_1 = V(C_n)_1 = \{a_1, a_2, a_3, \cdots, a_{n+1} = a_n\}$
- $V_2 = V(C_n)_2 = \{b_1, b_2, b_3, \cdots, b_{n+1} = b_n\}$

and let $E(G) = E_A \cup E_B \cup E_C$, where for $1 \leq x \leq n$

- $E_A = E(C_n)_1 = (a_x, a_{x+1})$
- $E_B = E(C_n)_2 = (b_x, b_{x+1})$
- $E_C = (a_x, b_x)$

Now, allocate the *RDC* to the vertices of G as follows:

Consider the set $C : \{c_1, c_2, \cdots, c_n\}$ of color classes as follows.

$$V(Y_n) = \begin{cases} c_x & \text{for } a_x, b_{x-1}, \text{ for } 1 \leq x \leq \lfloor \frac{n}{2} \rfloor \\ c_n & \text{for } a_n, b_1. \end{cases}$$

Here, for each $x = 1$, the vertices a_x, b_{x+1} of the color class c_x dominate the vertices a_n, b_1 of the color class c_n . The vertices a_n, b_1 of the color class c_n dominate the vertices a_x, b_{x+1} of the color class c_x for $x = 1$. Also, for $2 \leq x \leq n-1$, the vertices a_x, b_{x-1} of the color class c_x dominate any other vertices of the color class c_x .

Thus, $\chi_{rd}(Y_n) \leq n$. Now, to prove that $\chi_{rd}(Y_n) \geq n$, we assume that $\chi_{rd}(Y_n) = n-1$. Then $n-1$ colors must be allocated to the vertices of G for a RDC.

As G contains two copies of C_n and any copy of C_n requires $\lceil \frac{n}{2} \rceil$ colors for a proper rainbow coloring. Now, if the remaining $(n-1) - \lceil \frac{n}{2} \rceil$ colors are allocated to the remaining vertices of another cycle of Y_n, then a simple verification shows that there exist some paths (like the paths:(b_x, b_{x+1}) for1 $\leq x \leq n)$ in which two of its vertices are colored with the same colors.

This is a contradiction.

Therefore, $\chi_{rd}(Y_n) \geq n$.

Thus, $\chi_{rd}(Y_n) = n$.

□

CONCLUSION

In this paper, a new vertex coloring parameter called the rainbow dominator chromatic number of a graph is introduced. Two existing coloring concepts namely rainbow vertex coloring and dominator coloring are the basis for this parameter. The rainbow dominator chromatic number of some standard graph structures is determined.

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Received : October 30, 2020
Accepted : March 10, 2021