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ON r-DYNAMIC COLORING OF PARA-LINE GRAPH OF SOME STANDARD GRAPHS

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Abstract Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. The Subdivision graph $S(G)$ of a graph G is the graph acquired by inserting a new vertex into every edge of G. The para-line graph of G is the line graph of the subdivision graph of G, which is symbolized by $L(S(G))$. An r-dynamic coloring of a graph G is a proper coloring c of the vertices such that $|c(N(v))| \geq min\{r, d(v)\}$, for each $v \in V(G)$. The r-dynamic chromatic number of a graph G is the minimum k such that G has an r-dynamic coloring with k colors. In this paper, we acquired the r-dynamic chromatic number of para-line graph of the some standard graphs.

1 Introduction

The conception of r-dynamic chromatic number was first initiated by Montgomery [14]. It is also consider under the name r- hued [15], [16]. The r- dynamic coloring is a generalization of the vertex coloring for which $r = 1$. An r-dynamic coloring of a graph G is a proper coloring and it maps c from $V(G)$ to the set of colors such that (i) if $uv \in E(G)$, then $c(u) \neq c(v)$, and (ii) for each vertex $v \in V(G)$, $|c(N(v))| \geq min\{r, d(v)\}$, where $N(v)$ denotes the set of vertices adjacent to v, $d(v)$ its degree and r is a positive integer. The r-dynamic chromatic number of a graph G, written $\chi_r(G)$, is the minimum k such that G has an r-dynamic proper k-coloring. In this paper we speculate only the graphs which are simple, finite, loopless and connected. For all terms and definition which are not precisely described in this paper, we cite to [3]. The r-dynamic chromatic number has been studied by many researcher, specifically in [1], [2], [4], [6], [7], [8], [9], [10], [11], [12], [13], [17].

2 Preliminaries

Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. The Subdivision graph $S(G)$ of a graph G is the graph obtained by inserting a new vertex into every edge of G. The line graph [5] of G denoted by $L(G)$ is the graph whose vertex set is the edge set of G. Two vertices of $L(G)$ are adjacent whenever the corresponding edges of G are adjacent. The para-line graph [18] of G is the line graph of the subdivision graph of G, which is denoted by $L(S(G))$. An r-dynamic coloring of a graph G is a proper coloring c of the vertices such that $|c(N(v))| \geq min\{r, d(v)\}$, for each $v \in V(G)$. Para-line graphs are requisitioned in structural chemistry.

3 Results

Lemma 3.1. *[13]* $\chi_r(G) \geq min\{r, \Delta(G)\} + 1$

In this section, we determine the r-dynamic chromatic number of para-line graph of path, cycle, complete graph, complete bipartite graph, fan graph, bistar graph, tadpole graph and lollipop graph. Firstly we will find the lower bounds of r-dynamic chromatic number of the graphs and we prove our theorems.

Lemma 3.2. *Let $L(S(P_n))$ be the Para-line graph of a Path graph P_n. The lower bound of r-Dynamic Chromatic number of $L(S(P_n))$ is*

$$\chi_r(L(S(P_n))) \geq \begin{cases} 2; & r = 1, \\ 3; & r \geq 2. \end{cases}$$

Proof. The vertex and edge set of Path graph is represented as follows.

$$V(P_n) = \{u_i : 1 \leq i \leq n\}.$$
$$E(P_n) = \{u_i u_{i+1} : 1 \leq i \leq n-1\}.$$

Let $V(S(P_n)) = \{u_i, v_j : 1 \leq i \leq n, 1 \leq j \leq n-1\}$ where v_i are the new vertices inserted on the edge $u_i u_{i+1}$ of P_n. The vertex set of the graph $L(S(P_n))$ is represented as $V(L(S(P_n))) = \{e_{ii}, f_{i(i+1)} : 1 \leq i \leq n-1\}$, where e_{ii} are the vertices corresponding to the edge $u_i v_i$ and f_{ij} are the vertices corresponding to the edge $v_i u_j$.
For $r = 1$, based on the Lemma 2.1, we have $\chi_r(G) \geq min\{r, \Delta(G)\} + 1$ such that $\chi_r(L(S(P_n))) \geq min\{r, \Delta(L(S(P_n)))\} + 1 = r + 1 = 2$.
For $r \geq 2$, from the Lemma 2.1, we obtain $\chi_r(L(S(P_n))) \geq min\{r, \Delta(L(S(P_n)))\} + 1 = \Delta(L(S(P_n))) + 1 = 2 + 1 = 3$. It concludes the proof. □

Theorem 3.3. *Let $n \geq 2$, the r-Dynamic Chromatic number of $L(S(P_n))$ is*

$$\chi_{r=1}(L(S(P_n))) = 2.$$
$$\chi_{r\geq 2}(L(S(P_n))) = 3.$$

Proof. The maximum and the minimum degrees of the graph $L(S(P_n))$ are obtained as $\Delta(L(S(P_n))) = 2$ and $\delta(L(S(P_n))) = 1$, respectively.
Case 1: $r = 1$
Proceeding from the Lemma 3.1, the lower bound is

$$\chi_r(L(S(P_n))) \geq 2.$$

To exhibit the upper bound, we describe a map $c : V(L(S(P_n))) \rightarrow \{c_1, c_2\}$ as follows.

$$\text{c}(\text{e}_{11}, f_{12}, e_{22}, f_{23}, \ldots, e_{(n-1)(n-1)}, f_{(n-1)n}) = \{c_1, c_2, c_1, c_2, \ldots\}$$

It is easy to explicit that c is a r-Dynamic Coloring.
Hence, $\chi_{r=1}(L(S(P_n))) \leq 2$.
Thus, $\chi_{r=1}(L(S(P_n))) = 2$.
Case 2: $r \geq 2$
Proceeding from the Lemma 3.1, the lower bound is

$$\chi_r(L(S(P_n))) \geq 3.$$

To exhibit the upper bound, we describe a map $c : V(L(S(P_n))) \rightarrow \{c_1, c_2, c_3\}$ as follows.

$$c(e_{11}, f_{12}, e_{22}, f_{23}, \ldots, e_{(n-1)(n-1)}, f_{(n-1)n}) = \{c_1, c_2, c_3, c_1, c_2, c_3, \ldots\}$$

It is easy to explicit that c is a r-Dynamic Coloring.
Hence, $\chi_r(L(S(P_n))) \leq 3$.
Thus, $\chi_{r\geq}(L(S(P_n))) = 3$. It conforms the proof. □

Lemma 3.4. *Let $L(S(C_n))$ be the Para-line graph of a Cycle graph C_n.
The lower bound of r-Dynamic Chromatic number of $L(S(C_n))$ is*

$$\chi_r(L(S(C_n))) \geq \begin{cases} 2; & r = 1, \\ 3; & r \geq 2. \end{cases}$$

Proof. The vertex and edge set of Cycle graph is represented as follows.

$$V(C_n) = \{u_i : 1 \leq i \leq n\}.$$
$$E(C_n) = \{u_i u_{i+1}, u_n u_1 : 1 \leq i \leq n-1\}.$$

Let $V(S(C_n)) = \{u_i, v_i : 1 \leq i \leq n\}$ where $\{v_i : 1 \leq i \leq n-1\}$ are the new vertices inserted on the edge $u_i u_{i+1}$ and v_n is the new vertex inserted on the edge $u_n u_1$ of C_n. Let $V(L(S(C_n))) = \{e_{ii}, f_{i(i+1)} : 1 \leq i \leq n-1\} \cup \{e_{nn}, f_{n1}\}$, where e_{ii} are the vertices corresponding to the edge $u_i v_i$ and f_{ij} are the vertices corresponding to the edge $v_i u_j$.
For $r = 1$, from the Lemma 2.1, we have $\chi_r(G) \geq min\{r, \Delta(G)\} + 1$ such that $\chi_r(L(S(C_n))) \geq min\{r, \Delta(L(S(C_n)))\} + 1 = r + 1 = 2$.
For $r \geq 2$, from the Lemma 2.1, we obtain $\chi_r(L(S(C_n))) \geq min\{r, \Delta(L(S(C_n)))\} + 1 = \Delta(L(S(C_n))) + 1 = 2 + 1 = 3$. It concludes the proof. □

Theorem 3.5. *Let $n \geq 3$, the r-Dynamic Chromatic number of $L(S(C_n))$ is*

$$\chi_{r=1}(L(S(C_n))) = 2.$$
$$\chi_{r\geq 2}(L(S(C_n))) = \begin{cases} 3; & 2n \equiv 0 (mod\, 3), \\ 4; & 2n \equiv 1, 2 (mod\, 3). \end{cases}$$

Proof. The maximum and the minimum degrees of the graph $L(S(C_n))$ are obtained as $\Delta(L(S(C_n))) = \delta(L(S(C_n))) = 2$.
Case 1: $r = 1$
In reference to the Lemma 3.3, the lower bound is

$$\chi_r(L(S(C_n))) \geq 2.$$

To exhibit the upper bound, we describe a map $c : V(L(S(C_n))) \to \{c_1, c_2\}$ as follows.

$$c(e_{11}, f_{12}, e_{22}, f_{23}, \ldots, e_{nn}, f_{n1}) = \{c_1, c_2, c_1, c_2, \ldots\}$$

It is easy to explicit that c is a r-Dynamic Coloring.
Hence, $\chi_r(L(S(C_n))) \leq 2$.
Thus, $\chi_{r=1}(L(S(C_n))) = 2$.
Case 2: $r \geq 2$
Subcase(i): $2n \equiv 0\ (mod\, 3)$
In reference to the Lemma 3.3, the lower bound is

$$\chi_r(L(S(C_n))) \geq 3.$$

To expose the upper bound, we describe a map $c : V(L(S(C_n))) \to \{c_1, c_2, c_3\}$ as follows.

$$c(e_{11}, f_{12}, e_{22}, f_{23}, \ldots, e_{nn}, f_{n1}) = \{c_1, c_2, c_3, c_1, c_2, c_3, \ldots\}$$

It is easy to explicit that c is a r-Dynamic Coloring.
Hence, $\chi_r(L(S(C_n))) \leq 3$.
Thus, $\chi_{r=2}(L(S(C_n))) = 3$, $2n \equiv 0\ (mod\, 3)$.
Subcase(ii): $2n \equiv 1, 2\ (mod\, 3)$
In reference to the Lemma 3.3, the lower bound is

$$\chi_r(L(S(C_n))) \geq 3.$$

To exhibit the upper bound, we describe a map $c : V(L(S(C_n))) \to \{c_1, c_2, c_3, c_4\}$ as follows.
For $2n \equiv 1 (mod\, 3)$,

$$c(e_{11}, f_{12}, \ldots, e_{nn}, f_{n1}) = \{c_1, c_2, c_3, c_1, c_2, c_3, \ldots, c_1, c_2, c_3, c_4\}$$

For $2n \equiv 2 (mod\, 3)$,

$$c(e_{11}, f_{12}, \ldots, e_{nn}, f_{n1}) = \{c_1, c_2, c_3, c_4, c_1, c_2, c_3, c_4, c_1, c_2, c_3, \ldots, c_1, c_2, c_3\}$$

It is easy to explicit that c is a r-Dynamic Coloring.
Hence, $\chi_r(L(S(C_n))) \leq 4$.
Thus, $\chi_{r\geq 2}(L(S(C_n))) = 4$, $2n \equiv 1, 2\ (mod\, 3)$. It conforms the proof. □

Lemma 3.6. *Let $L(S(K_n))$ be the Para-line graph of a Complete graph K_n. The lower bound of r-Dynamic Chromatic number of $L(S(K_n))$ is*

$$\chi_r(L(S(K_n))) \geq \begin{cases} n-1; & 1 \leq r \leq \Delta(L(S(K_n))) - 1, \\ \Delta+1; & r \geq \Delta(L(S(K_n))). \end{cases}$$

Proof. The vertex and edge set of Complete graph is represented as follows.

$$V(K_n) = \{u_i : 1 \leq i \leq n\}.$$
$$E(K_n) = \{u_i u_j : 1 \leq i, j \leq n, i < j\}.$$

Let $V(S(K_n)) = \{u_i, v_{ij} : 1 \leq i, j \leq n, i < j\}$ where v_{ij} are the new vertices inserted on the edge $u_i u_j$ of K_n. The vertex set of the graph $L(S(C_n))$ is represented as
$V(L(S(K_n))) = \{e_{i,ij}, e_{j,ij} : 1 \leq i \leq n-1, 1 \leq j \leq n, i \neq j\}$, where $e_{i,ij}$ are the vertices corresponding to the edges $u_i v_{ij} (i < j)$ and $e_{j,ij}$ are the vertices corresponding to the edges $u_j v_{ij} (i > j)$.
Clearly the vertices $\{e_{i,ij} : 1 \leq i \leq n, 2 \leq j \leq n\}$ induces a clique of order K_{n-1} in $L(S(K_n))$. For $1 \leq r \leq \Delta(L(S(K_n))) - 1$, $\chi_r(L(S(K_n))) \geq n-1$. For $r \geq \Delta(L(S(K_n)))$, based on the Lemma 2.1, we obtain $\chi_r(L(S(K_n))) \geq min\{r, \Delta(L(S(K_n)))\} + 1 = \Delta(L(S(K_n))) + 1$. It concludes the proof. □

Theorem 3.7. *Let $n \geq 6$, the r-Dynamic Chromatic number of $L(S(K_n))$ is*

$$\chi_r(L(S(K_n))) = \begin{cases} n-1; & 1 \leq r \leq \Delta - 1, \\ n; & r \geq \Delta. \end{cases}$$

Proof. The maximum and the minimum degrees of the graph $L(S(K_n))$ are obtained as
$\Delta(L(S(K_n))) = \delta(L(S(K_n))) = n-1$.
Case 1: $1 \leq r \leq \Delta - 1$
In reference to the Lemma 3.5, the lower bound is

$$\chi_r(L(S(K_n))) \geq n-1.$$

To exhibit the upper bound, we describe a map $c : V(L(S(K_n))) \rightarrow \{c_1, c_2, \ldots, c_{n-1}\}$ as follows.

$$c(e_{1,12}, e_{1,13}, \ldots, e_{1,1n}) = \{c_1, c_2, \ldots, c_{n-1}\}$$
$$c(e_{2,12}, e_{2,23}, \ldots, e_{2,2n}) = \{c_2, c_3, \ldots, c_{n-1}, c_1\}$$

Proceeding in the same manner we define,

$$c(e_{n-1,1(n-1)}, e_{n-1,2(n-1)}, \ldots, e_{n-1,(n-1)n}) = \{c_{n-1}, c_1, c_2, \ldots, c_{n-2}\}$$
$$c(e_{n,1n}, e_{n,2n}, \ldots, e_{n,(n-1)n}) = \{c_1, c_2, \ldots, c_{n-1}\}$$

It is easy to explicit that c is a r-Dynamic Coloring.
Hence, $\chi_r(L(S(K_n))) \leq n-1$.
Thus, $\chi_{1 \leq r \leq \Delta - 1}(L(S(K_n))) = n-1$.
Case 2: $r \geq \Delta$
In reference to the Lemma 3.5, the lower bound is

$$\chi_r(L(S(K_n))) \geq n.$$

To exhibit the upper bound, we describe a map $c : V(L(S(K_n))) \rightarrow \{c_1, c_2, \ldots, c_n\}$ as follows.

$$c(e_{1,12}, e_{1,13}, \ldots, e_{1,1n}) = \{c_1, c_2, \ldots, c_{n-1}\}$$
$$c(e_{2,12}, e_{2,23}, \ldots, e_{2,2n}) = \{c_n, c_2, c_3, \ldots, c_{n-1}\}$$
$$c(e_{3,13}, e_{3,23}, \ldots, e_{3,3n}) = \{c_n, c_1, c_3, \ldots, c_{n-1}\}$$

Proceeding in the same manner we define,

$$c(e_{n,1n}, e_{n,2n}, \ldots, e_{n,(n-1)n}) = \{c_n, c_1, c_2, \ldots, c_{n-2}\}$$

It is easy to explicit that c is a r-Dynamic Coloring.
Hence, $\chi_r(L(S(K_n))) \leq n$.
Thus, $\chi_{r\geq\Delta}(L(S(K_n))) = n$. It conforms the proof. □

Lemma 3.8. *Let $L(S(K_{m,n}))$ be the Para-line graph of a Complete bipartite graph $K_{m,n}$. The lower bound of r-Dynamic Chromatic number of $L(S(K_{m,n}))$ is*

$$\chi_r(L(S(K_{m,n}))) \geq \begin{cases} n; & 1 \leq r \leq \Delta(L(S(K_{m,n}))) - 1, \\ \Delta + 1; & r \geq \Delta(L(S(K_{m,n}))). \end{cases}$$

Proof. The vertex and edge set of Complete bipartite graph is represented as follows.

$$V\left(K_{m,n}\right) = \left\{x_i, y_j : 1 \leq i \leq m, m+1 \leq j \leq m+n\right\}.$$
$$E\left(K_{m,n}\right) = \left\{x_i y_j : 1 \leq i \leq m, m+1 \leq j \leq m+n\right\}.$$

Let $V\left(S(K_{m,n})\right) = \{x_i, y_j, v_{ij}\}$ where v_{ij} are the new vertices inserted on the edge $x_i y_j$ of $K_{m,n}$. The vertex set of the graph $L(S(K_{m,n}))$ is represented as
$V\left(L(S(K_{m,n}))\right) = \{e_{i,ij} \cup e_{j,ij} : 1 \leq i \leq m, m+1 \leq j \leq m+n\}$.
Clearly the vertices $\{e_{i,ij} : 1 \leq i \leq m, m+1 \leq j \leq m+n\}$ induces a clique of order K_n in $L(S(K_{m,n}))$.
For $1 \leq r \leq \Delta(L(S(K_{m,n}))) - 1$, we have $\chi_r(L(S(K_{m,n}))) \geq n$.
For $r \geq \Delta(L(S(K_{m,n})))$, based on the Lemma 2.1, we obtain
$\chi_r(L(S(K_{m,n}))) \geq min\{r, \Delta(L(S(K_{m,n})))\} + 1 = \Delta(L(S(K_{m,n}))) + 1$. It concludes the proof. □

Theorem 3.9. *Let $m, n \geq 3$, $m < n$, the r-Dynamic Chromatic number of $L(S(K_{m,n}))$ is*

$$\chi_r(L(S(K_{m,n}))) = \begin{cases} n; & 1 \leq r \leq \Delta - 1, \\ m+n; & r \geq \Delta. \end{cases}$$

Proof. The maximum and the minimum degrees of the graph $L(S(K_{m,n}))$ are obtained as $\Delta(L(S(K_{m,n}))) = n$ and $\delta(L(S(K_{m,n}))) = m$.
Case 1: $1 \leq r \leq \Delta - 1$
In reference to the Lemma 3.7, the lower bound is

$$\chi_r(L(S(K_{m,n}))) \geq n.$$

To exhibit the upper bound, we describe a map $c : V\left(L(S(K_{m,n}))\right) \to \{c_1, c_2, \ldots, c_n\}$ as follows.

$$c(e_{i,i(m+1)}, e_{i,i(m+2)}, \ldots, e_{i,i(m+n)}) = \{c_1, c_2, \ldots, c_n\}, \ for\ 1 \leq i \leq m.$$
$$c(e_{(m+1),1(m+1)}, e_{(m+1),2(m+1)}, \ldots, e_{(m+1),m(m+1)}) = \{c_2, c_3, \ldots, c_{m+1}\}$$
$$c(e_{(m+2),1(m+2)}, e_{(m+2),2(m+2)}, \ldots, e_{(m+2),m(m+2)}) = \{c_1, c_3, c_4, \ldots, c_{m+1}\}$$

Proceeding in the same manner we define,

$$c(e_{(m+n),1(m+n)}, e_{(m+n),2(m+n)}, \ldots, e_{(m+n),m(m+n)}) = \{c_1, c_2, \ldots, c_m\}$$

It is easy to explicit that c is a r-Dynamic Coloring.
Hence, $\chi_r(L(S(K_{m,n}))) \leq n$.
Thus, $\chi_{1\leq r\leq\Delta-1}(L(S(K_{m,n}))) = n$.
Case 2: $r \geq \Delta$
From the the Lemma 3.7, the lower bound is

$$\chi_r(L(S(K_{m,n}))) \geq \Delta + 1 = n + 1.$$

To exhibit the upper bound, we describe a map $c : V\left(L(S(K_{m,n}))\right) \to \{c_1, c_2, \ldots, c_{m+n}\}$ as follows.

$$c(e_{i,i(m+1)}, e_{i,i(m+2)}, \ldots, e_{i,i(m+n)}) = \{c_1, c_2, \ldots, c_n\}, \ for\ 1 \leq i \leq m.$$
$$c(e_{j,1j}, e_{j,2j}, \ldots, e_{j,mj}) = \{c_{m+1}, c_{m+2}, \ldots, c_{m+n}\}, \ for\ m+1 \leq j \leq m+n.$$

It is easy to explicit that c is a r-Dynamic Coloring.
Hence, $\chi_r(L(S(K_{m,n}))) \leq m+n$.
Thus, $\chi_{r\geq\Delta-1}(L(S(K_{m,n}))) = m+n$.
It completes the proof. □

Lemma 3.10. *Let $L(S(B_{m,n}))$ be the Para-line graph of a Bistar graph $B_{m,n}$. The lower bound of r-Dynamic Chromatic number of $L(S(B_{m,n}))$ is*

$$\chi_r(L(S(B_{m,n}))) \geq \begin{cases} n+1; & 1 \leq r \leq \Delta(L(S(B_{m,n}))) - 1, \\ \Delta+1; & r \geq \Delta(L(S(B_{m,n}))). \end{cases}$$

Proof. The vertex and edge set of Bistar graph is represented as follows.

$$V(B_{m,n}) = \{u_i, v_j : 1 \leq i \leq m+1, 1 \leq j \leq n+1\}.$$
$$E(B_{m,n}) = \{u_1v_1, u_1u_i, v_1v_j : 2 \leq i \leq m, 2 \leq j \leq n\}.$$

The vertex set of subdivision graph of Bistar graph is represented as
$V(S(B_{m,n})) = \{u_i, v_j, u_{1j}, v_{1k}, w_{11}\}$ where u_{1j} are the new vertices inserted on the edge u_1u_j, v_{1j} are the new vertices inserted on the edge v_1v_j and w_{11} is the new vertex inserted on the edge u_1v_1 of $B_{m,n}$.
The vertex set of Para-line graph of Bistar graph $L(S(B_{m,n}))$ is represented as
$V(L(S(B_{m,n}))) = \{e_{1,1j} : 1 \leq j \leq m+1\} \cup \{e_{j,1j} : 2 \leq j \leq m+1\}$
$\cup \{f_{1,1k} : 1 \leq k \leq n+1\} \cup \{f_{k,1k} : 2 \leq k \leq n+1\}$, where $e_{1,11}$ is the vertex corresponding to the edges u_1w_{11}, $e_{1,1j}$ are the vertices corresponding to the edges u_1u_{1j}, $e_{j,1j}$ are the vertices corresponding to the edges u_ju_{1j}. Similarly $f_{1,11}$ is the vertex corresponding to the edges v_1w_{11}, $f_{1,1j}$ are the vertices corresponding to the edges v_1v_{1j}, $f_{j,1j}$ are the vertices corresponding to the edges v_jv_{1j}.
For $1 \leq r \leq \delta(L(S(B_{m,n})))$, the vertices $V = \{f_{1,1j} : 2 \leq j \leq n+1\}$ induce a clique of order $K_n + 1$ in $L(S(B_{m,n}))$. Thus, we have $\chi_r(L(S(B_{m,n}))) \geq n+1$.
For $r \geq \Delta(L(S(B_{m,n})))$, based on the Lemma 2.1, we obtain
$\chi_r(L(S(B_{m,n}))) \geq min\{r, \Delta(L(S(B_{m,n})))\} + 1 = \Delta(L(S(B_{m,n}))) + 1$.
It concludes the proof. □

Theorem 3.11. *Let $m \geq 3$, $m < n$, the r-Dynamic Chromatic number of $L(S(B_{m,n}))$ is*

$$\chi_r(L(S(B_{m,n}))) = \begin{cases} n+1; & 1 \leq r \leq \Delta - 1, \\ n+2; & r \geq \Delta. \end{cases}$$

Proof. The maximum and the minimum degrees of the graph $L(S(B_{m,n}))$ are obtained
$\Delta(L(S(B_{m,n}))) = n+1$ and $\delta(L(S(B_{m,n}))) = 1$.
Case 1: $1 \leq r \leq \Delta - 1$
In reference to the Lemma 3.9, the lower bound is

$$\chi_r(L(S(B_{m,n}))) \geq n+1.$$

To exhibit the upper bound, we describe a map $c : V(L(S(B_{m,n}))) \to \{c_1, c_2, \ldots, c_{n+1}\}$ as follows.

$$c(f_{1,1j}) = c_j, \; for \; 1 \leq j \leq n+1.$$
$$c(e_{1,1j}) = c_{j+1}, \; for \; 1 \leq j \leq m+1.$$
$$c(e_{i,1i}) = c_1, \; for \; 2 \leq i \leq m+1.$$
$$c(f_{j,1j}) = c_1, \; 2 \leq j \leq n+1.$$

For $2 \leq i \leq m+1$ and $2 \leq j \leq n+1$,
It is easy to explicit that c is a r-Dynamic Coloring.
Hence, $\chi_r(L(S(B_{m,n}))) \leq n+1$.
Thus, $\chi_{1\leq r\leq\Delta-1}(L(S(B_{m,n}))) = n+1$.
Case 2: $r \geq \Delta$
In reference to the Lemma 3.9, the lower bound is

$$\chi_r(L(S(B_{m,n}))) \geq n+2.$$

To exhibit the upper bound, we describe a map $c : V\left(L(S(B_{m,n}))\right) \to \{c_1, c_2, \ldots, c_{n+2}\}$ as follows.

$$\begin{aligned} c(f_{1,1j}) &= c_j,\ for\ 1 \leq j \leq n+1. \\ c(e_{1,11}, e_{1,12}, \ldots, e_{1,1(m+1)}) &= \{c_{n+2}, c_2, c_3, \ldots, c_{m+1}\} \\ c(e_{i,1i}) &= c_1,\ for\ 2 \leq i \leq m+1. \\ c(f_{j,1j}) &= c_{n+2},\ for\ 2 \leq j \leq n+1. \end{aligned}$$

It is easy to explicit that c is a r-Dynamic Coloring.
Hence, $\chi_r(L(S(B_{m,n}))) \leq n+2$.
Thus, $\chi_{r\geq\Delta}(L(S(B_{m,n}))) = n+2$.
It completes the proof. □

Lemma 3.12. *Let $L(S(T_{m,n}))$ be the Para-line graph of a Tadpole graph $T_{m,n}$. The lower bound of r-Dynamic Chromatic number of $L(S(T_{m,n}))$ is*

$$\chi_r(L(S(T_{m,n}))) \geq \begin{cases} 3; & r = 1 and\ 2, \\ 4; & r \geq 3. \end{cases}$$

Proof. The vertex and edge set of Tadpole graph is represented as follows.

$$\begin{aligned} &V\left(T_{m,n}\right) = \left\{u_i : 1 \leq i \leq m+n\right\}. \\ &E\left(T_{m,n}\right) = \left\{u_i u_{i+1}, u_m u_1, u_j u_{j+1} : 1 \leq i \leq m-1, m \leq j \leq m+n-1\right\}. \end{aligned}$$

Let $V\left(S(T_{m,n})\right) = \{u_i, v_{ij}, i < j\}$ where v_{ij} are the new vertices inserted on the edge $u_i u_j$. The vertex set of Para-line graph of Tadpole graph $L(S(T_{m,n}))$ is represented as
$V\left(L(S(T_{m,n}))\right) = \{e_{i,jk}\}$, where $e_{i,jk}$ are the vertex corresponding to the edges $u_i v_{jk}$ of $T_{m,n}$.
For $r = 1$ and 2, the vertices $V = \{e_{m,(m-1)m}, e_{m,1m}, e_{m,m(m+1)}\}$ induce a clique of order K_3 in $L(S(T_{m,n}))$. Thus, $\chi_r(L(S(T_{m,n}))) \geq 3$.
For $r \geq 3$, based on the Lemma 2.1, we obtain
$\chi_r(L(S(T_{m,n}))) \geq min\{r, \Delta(L(S(T_{m,n})))\} + 1 = \Delta(L(S(T_{m,n}))) + 1 = 3+1 = 4$. It concludes the proof. □

Theorem 3.13. *Let $m > 4, n \geq 3$, the r-Dynamic Chromatic number of $L(S(T_{m,n}))$ is*

$$\begin{aligned} \chi_{r=1,2}(L(S(T_{m,n}))) &= 3. \\ \chi_{r\geq 3}(L(S(T_{m,n}))) &= 4. \end{aligned}$$

Proof. The maximum and the minimum degrees of the graph $L(S(T_{m,n}))$ are obtained as $\Delta(L(S(T_{m,n}))) = 3$ and $\delta(L(S(T_{m,n}))) = 1$.
Case 1: $r = 1$ and 2
In reference to the Lemma 3.11, the lower bound is

$$\chi_r(L(S(T_{m,n}))) \geq 3.$$

To exhibit the upper bound, we describe a map $c : V\left(L(S(T_{m,n}))\right) \to \{c_1, c_2, c_3\}$ as follows.

$$\begin{aligned} c(e_{1,12}, e_{2,12}, \ldots, e_{m,(m-1)m}, e_{m,1m}, e_{1,1m}) &= \{c_1, c_2, c_1, c_2, \ldots, c_1, c_2\} \\ c(e_{m,m(m+1)}, e_{m+1,m(m+1)}, \ldots, e_{(m+n),(m+n-1)(m+n)}) &= \{c_3, c_1, c_2, c_1, c_2, \ldots, \} \end{aligned}$$

It is easy to explicit that c is a r-Dynamic Coloring.
Hence, $\chi_r(L(S(T_{m,n}))) \leq 3$.
Thus, $\chi_{r=1,2}(L(S(T_{m,n}))) = 3$.
Case 2: $r \geq 3$
In reference to the Lemma 3.11, the lower bound is

$$\chi_r(L(S(T_{m,n}))) \geq 4.$$

To exhibit the upper bound, we describe a map $c : V\left(L(S(T_{m,n}))\right) \rightarrow \{c_1, c_2, c_3, c_4\}$ as follows.

$$c(e_{1,12}, e_{2,12}, \ldots, e_{m,1m}, e_{1,1m}) = \begin{cases} c_1, c_2, c_3, c_1, c_2, c_3, \ldots, c_1, c_2, c_3; & 2n \equiv 0\ (mod\ 3) \\ c_1, c_2, c_3, c_4, c_1, c_2, c_3, \ldots, c_1, c_2, c_3; & 2n \equiv 1\ (mod\ 3) \end{cases}$$

For $2n \equiv 2(mod\ 3)$

$$c(e_{1,12}, e_{2,12}, \ldots, e_{m,1m}, e_{1,1m}) = \{c_1, c_2, c_3, c_4, c_1, c_2, c_3, c_4, c_1, c_2, c_3, \ldots, c_1, c_2, c_3\}$$

For $2n \equiv 0(mod\ 3)$

$$\text{c}(\text{e}_{m,m(m+1)}, e_{m+1,m(m+1)}, \ldots, e_{(m+n),(m+n-1)(m+n)}) = \{c_4, c_3, c_2, c_4, c_3, c_2, \ldots\}$$

For $2n \equiv 1, 2(mod\ 3)$

$$\text{c}(\text{e}_{m,m(m+1)}, e_{m+1,m(m+1)}, \ldots, e_{(m+n),(m+n-1)(m+n)}) = \{c_4, c_1, c_2, c_4, c_1, c_2, \ldots\}$$

It is easy to explicit that c is a r-Dynamic Coloring.
Hence, $\chi_r(L(S(T_{m,n}))) \leq 4$.
Thus, $\chi_{r\geq 3}(L(S(T_{m,n}))) = 4$. It conforms the proof. □

Lemma 3.14. *Let $L(S(L_{m,n}))$ be the Para-line graph of a Lollipop graph $L_{m,n}$. The lower bound of r-Dynamic Chromatic number of $L(S(L_{m,n}))$ is*

$$\chi_r(L(S(L_{m,n}))) \geq \begin{cases} m; & 1 \leq r \leq \Delta(L(S(L_{m,n}))) - 1, \\ \Delta + 1; & r \geq \Delta(L(S(L_{m,n}))). \end{cases}$$

Proof. The vertex and edge set of Lollipop graph is represented as follows.

$$V\left(L_{m,n}\right) = \{u_i : 1 \leq i \leq m+n\}.$$
$$E\left(L_{m,n}\right) = \{u_iu_j : 1 \leq i, j \leq m, i < j\} \cup \{u_iu_{i+1} : m \leq i \leq m+n-1\}.$$

Let $V\left(S(L_{m,n})\right) = \{u_i, v_{ij}, i < j\}$ where v_{ij} are the new vertices inserted on the edge u_iu_j of $L_{m,n}$. The vertex set of Para-line graph of Lollipop graph $L(S(L_{m,n}))$ is represented as $V\left(L(S(L_{m,n}))\right) = \{e_{i,jk}\}$, where $e_{i,jk}$ are the vertex corresponding to the edges u_iv_{jk}.
For $1 \leq r \leq \delta(L(S(L_{m,n})) - 1$, the vertices $V = \{e_{m,im}, e_{m,m(m+1)} : 1 \leq i \leq m-1\}$ induce a clique of order K_m in $L(S(L_{m,n}))$. Thus, $\chi_r(L(S(L_{m,n}))) \geq m$.
For $r \geq \Delta(L(S(L_{m,n})))$, based on the Lemma 2.1, we obtain
$\chi_r(L(S(L_{m,n}))) \geq min\,\{r, \Delta(L(S(L_{m,n})))\} + 1 = \Delta(L(S(L_{m,n}))) + 1$. It concludes the proof. □

Theorem 3.15. *Let $m \geq 5, n \geq 3$, the r-Dynamic Chromatic number of $L(S(L_{m,n}))$ is*

$$\chi_r(L(S(L_{m,n}))) = \begin{cases} m; & 1 \leq r \leq \Delta(L(S(L_{m,n}))) - 1, \\ m+1; & r \geq \Delta(L(S(L_{m,n}))). \end{cases}$$

Proof. The maximum and the minimum degrees of the graph $L(S(L_{m,n}))$ are obtained as $\Delta(L(S(L_{m,n}))) = m$ and $\delta(L(S(L_{m,n}))) = 1$.
Case 1: $1 \leq r \leq \Delta(L(S(L_{m,n}))) - 1$
In reference to the Lemma 3.13, the lower bound is

$$\chi_r(L(S(L_{m,n}))) \geq m.$$

To exhibit the upper bound, we describe a map $c : V\left(L(S(L_{m,n}))\right) \rightarrow \{c_1, c_2, \ldots, c_m\}$ as follows.

$$c(e_{1,12}, e_{1,13}, \ldots, e_{1,1m}) = \{c_1, c_2, \ldots, c_{m-1}\}$$
$$c(e_{2,12}, e_{2,23}, \ldots, e_{2,2m}) = \{c_2, c_3, \ldots, c_{m-1}, c_1\}$$

Proceeding in the same manner finally we define,

$$c(e_{m,1m}, e_{m,2m}, \ldots, e_{m,(m-1)m}) = \{c_1, c_2, \ldots, c_{m-1}\}$$
$$c(e_{m,m(m+1)}, e_{m+1,m(m+1)}, \ldots, e_{(m+n),(m+n-1)(m+n)}) = \{c_m, c_1, c_2, c_m, c_1, c_2, \ldots,\}$$

It is easy to explicit that c is a r-Dynamic Coloring.
Hence, $\chi_r(L(S(L_{m,n}))) \leq m$. Thus, $\chi_{1\leq r\leq \Delta-1}(L(S(L_{m,n}))) = m$.
Case 2: $r \geq \Delta$
In reference to the Lemma 3.13, the lower bound is

$$\chi_r(L(S(L_{m,n}))) \geq m+1.$$

To exhibit the upper bound, we describe a map $c : V(L(S(L_{m,n})))) \to \{c_1, c_2, \ldots, c_{m+1}\}$ as follows.

$$c(e_{1,12}, e_{1,13}, \ldots, e_{1,1m}) = \{c_1, c_2, \ldots, c_{m-1}\}$$
$$c(e_{2,12}, e_{2,23}, \ldots, e_{2,2m}) = \{c_m, c_2, c_3, \ldots, c_{m-1}\}$$

Proceeding in the same manner finally we define

$$c(e_{m,1m}, e_{m,2m}, \ldots, e_{m,(m-1)m}) = \{c_m, c_1, c_2, \ldots, c_{m-2}\}$$
$$c(e_{m,m(m+1)}, e_{m+1,m(m+1)}, \ldots, e_{(m+n),(m+n-1)(m+n)}) = \{c_{m+1}, c_{m-1}, c_1, c_{m+1}, c_{m-1}, c_1, \ldots\}$$

It is easy to explicit that c is a r-Dynamic Coloring.
Hence, $\chi_r(L(S(L_{m,n}))) \leq m+1$. Thus, $\chi_{r\geq\Delta}(L(S(L_{m,n}))) = m+1$. □

Lemma 3.16. *Let $L(S(F_{1,n}))$ be the Para-line graph of a Fan graph $F_{1,n}$. The lower bound of r-Dynamic Chromatic number of $L(S(F_{1,n}))$ is*

$$\chi_r(L(S(F_{1,n}))) \geq \begin{cases} n; & 1 \leq r \leq \Delta - 1, \\ \Delta + 1; & r \geq \Delta. \end{cases}$$

Proof. The vertex and edge set of Fan graph is represented as follows.

$$V(F_{1,n}) = \{u_i : 1 \leq i \leq n+1\}.$$
$$E(F_{1,n}) = \{u_1 u_i : 2 \leq i \leq n+1\} \cup \{u_i u_{i+1} : 2 \leq i \leq n-1\}.$$

Let $V(S(F_{1,n})) = \{u_i, v_{ij}, i < j\}$ where v_{ij} are the new vertices inserted on the edge $u_i u_j$ of $F_{1,n}$. The vertex set of Para-line graph of Fan graph $L(S(F_{1,n}))$ is represented as $V(L(S(F_{1,n}))) = \{e_{1,1j}, e_{j,1j} : 2 \leq j \leq n+1\} \cup \{e_{j,j(j+1)}, e_{(j+1),j(j+1)} : 2 \leq j \leq n\}$.
For $1 \leq r \leq \delta(L(S(F_{1,n}))$, the vertices $V = \{e_{1,1j} : 2 \leq j \leq n+1\}$ induce a clique of order K_n in $L(S(F_{1,n}))$. Thus, $\chi_r(L(S(F_{1,n}))) \geq n$.
For $r \geq \Delta(L(S(F_{1,n})))$, based on the Lemma 2.1, we obtain
$\chi_r(L(S(F_{1,n}))) \geq min\{r, \Delta(L(S(F_{1,n})))\}+1 = \Delta(L(S(F_{1,n})))+1$. It concludes the proof. □

Theorem 3.17. *Let $n \geq 6$, the r-Dynamic Chromatic number of $L(S(F_{1,n}))$ is*

$$\chi_r(L(S(F_{1,n}))) = \begin{cases} n; & 1 \leq r \leq \Delta - 1, \\ n+1; & r \geq \Delta. \end{cases}$$

Proof. The maximum and the minimum degrees of the graph $L(S(F_{1,n}))$ are obtained as $\Delta(L(S(F_{1,n}))) = n$ and $\delta(L(S(F_{1,n}))) = 2$.
Case 1: $1 \leq r \leq \Delta(L(S(F_{1,n}))) - 1$
From the Lemma 3.15, the lower bound is

$$\chi_r(L(S(F_{1,n}))) \geq n.$$

To exhibit the upper bound, we describe a map $c : V(L(S(F_{1,n}))) \to \{c_1, c_2, \ldots, c_n\}$ as follows.

$$c(e_{1,12}, e_{1,13}, \ldots, e_{1,1(n+1)}) = \{c_1, c_2, \ldots, c_n\}$$
$$c(e_{j,1j}) = c_n, \ for\ 2 \leq j \leq n.$$
$$c(e_{n+1,1(n+1)}) = c_1$$
$$c(e_{2,23}, e_{3,23}, \ldots, e_{n,n(n+1)}, e_{(n+1),n(n+1)}) = \{c_2, c_3, \ldots, c_{n-1}, c_2, c_3, c_4, c_2, c_3, c_4, \ldots\}$$

It is easy to explicit that c is a r-Dynamic Coloring.
Hence, $\chi_r(L(S(F_{1,n}))) \leq n$.
Thus, $\chi_{1\leq r\leq \Delta-1}(L(S(F_{1,n}))) = n$.
Case 2: $r \geq \Delta(L(S(F_{1,n})))$
Based on the Lemma 3.15, the lower bound is

$$\chi_r(L(S(F_{1,n}))) \geq \Delta + 1 = n + 1.$$

To exhibit the upper bound, we describe a map $c : V(L(S(F_{1,n}))) \to \{c_1, c_2, \ldots, c_{n+1}\}$ as follows.

$$c(e_{1,12}, e_{1,13}, \ldots, e_{1,1(n+1)}) = \{c_1, c_2, \ldots, c_n\}$$
$$c(e_{j,1j}) = c_{n+1},\ for\ 2 \leq j \leq n+1.$$
$$c(e_{2,23}, e_{3,23}, \ldots, e_{n,n(n+1)}, e_{(n+1),n(n+1)}) = \{c_2, c_3, c_4, \ldots, c_n, c_1, c_2, c_3, c_1, c_2, c_3, \ldots\}$$

It is easy to explicit that c is a r-Dynamic Coloring.
Hence, $\chi_r(L(S(F_{1,n}))) \leq n + 1$.
Thus, $\chi_{r\geq\Delta}(L(S(F_{1,n}))) = n + 1$. It conforms the proof. □

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