ON EQUITABLE COLORING OF EXTENDED CORONA OF SOME GRAPHS

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Abstract A proper $h$–colorable graph $K$ is claimed to be equitably $h$–colorable if the vertex set of $K$ can be partitioned into $h$ independent color classes $V_1, V_2, ..., V_h$ such that the condition $\left| |V_i| - |V_j| \right| \leq 1$ holds for all different pairs of $(i', j')$. And the smallest integer $h$ is called the equitable chromatic number of $K$ [5, 7]. In this paper, we consider an equitable coloring of the extended corona product of two graphs $K$ and $H$. In particular, we study the cases where $K$ and $H$ are complete graphs, cycles and paths. We also discussed the probability mass function, chromatic mean, and chromatic variance of these graph products.

1 Introduction

Many real world situations can conveniently be described by means of a diagram consisting of set of points together with lines joining certain pairs of these points [1].

The syllabary of an equitable coloring was first initiated by W. Meyer in the year 1973 [8]. Tucker’s paper [14], in which nodes represent garbage collection routes and adjacency of two such vertices when the corresponding routes should not be run on the same day. Then Meyer come up with the solution that equal number of routes run on every day in a week.

The graph $K$ is equitably colored with $h$ colors, if the absolute difference between their color classes is atmost one[6, 9]. The smallest integer $h$ for which $K$ is equitably $h$-colorable is known as the equitable chromatic number of $K$ and denoted by $\chi_e(K)$. Since an equitable coloring is a proper vertex coloring, we have

$$\chi_e(K) \geq \chi(K),$$

where $\chi(K)$ is the chromatic number of graph $K$.

2 Preliminaries

The corona $K_1 \circ K_2$ of two graphs $K_1$ and $K_2$ (where $K_i$ has $p_i$ points and $q_i$ lines) is defined as the graph $K$ obtained by taking one copy of $K_1$ and $p_i$ copies of $K_2$, and then joining by a line the $i$-th point of $K_1$ to every point in the $i$-th copy of $K_2$ [4].

The extended corona $K_1 \bullet K_2$ of two graphs $K_1$ and $K_2$ is a graph obtained by taking the corona $K_1 \circ K_2$ and joining each vertex of $i$-th copy of $K_2$ to every vertex of $j$-th copy of $K_2$, provided the vertices $v_{i'}$ and $v_{j'}$ are adjacent in $K_1$ [2].

A path is a non-empty graph $P = (V(P), E(P))$ of the form $V(P) = \{a_1, a_2, ..., a_s\}$ and $E(P) = \{a_1a_2, a_2a_3, ..., a_{s-1}a_s\}$. The number vertices of a path is its length $[1, 3]$.

A cycle is a closed trail $r \geq 3$, for which each vertices are distinct(except, of course first and last) [3].

A complete graph $K_q$ has $q$ points and all two-element subsets of $K_q$ as edges [3, 10].

Let $\{c_1, c_2, c_3, ..., c_h\}$ be a set of colors used in a proper $h$-coloring $C$ of $K$ and let $\theta(c_i)$ denote the number of times a particular color $c_i$ is assigned to nodes of $K$ [6]. Let $X$ be a random variable $(r:v)$ which denotes the color of an arbitrarily chosen vertex in $K$. Since the
sum of all weights of colors of $K$ is the order of $K$, the real valued function $f(i)$ defined by

$$f(i) = \begin{cases} \frac{\theta(c_i)}{|V(K)|} & i = 1, 2, 3, \ldots, h; \\ 0 & \text{elsewhere} \end{cases}$$

is the probability mass function (p.m.f) of the $X$ [11, 12, 13]. Here we say $f(i)$ is the p.m.f of the graph $K$ with respect to the given coloring $C$.

The coloring mean [11, 12, 13] of a coloring $C$ of a given graph $K$ is denoted by $\mu_c(K)$ and defined to be

$$\mu_c(K) = \frac{\sum_{i=1}^{h} i \theta(c_i)}{\sum_{i=1}^{h} \theta(c_i)}$$

The coloring variance [11, 12, 13] of a coloring $C$ of a given graph $K$ is denoted by $\sigma_c^2(K)$ and defined to be

$$\sigma_c^2(K) = \frac{\sum_{i=1}^{h} i^2 \theta(c_i)}{\sum_{i=1}^{h} \theta(c_i)} - \left( \frac{\sum_{i=1}^{h} i \theta(c_i)}{\sum_{i=1}^{h} \theta(c_i)} \right)^2$$

The chromatic mean and variance corresponding to an equitable coloring of a graph $K$ is defined as follows [12]. A coloring mean of a graph $K$, with respect to a proper coloring $C$ is said to be an equitable chromatic mean or a $\chi_e$ chromatic mean of $K$, if $C$ is the optimal (in the sense of the number of colors) equitable coloring of $K$. The $\chi_e$-chromatic mean of a graph $K$ is denoted by $\mu_{\chi_e}(K)$.

3 Results

Theorem 3.1. The equitable coloring of extended corona of complete graph ($K_q \bullet K_p$) with complete graph ($K_p$) is given by

$$\chi_e = (K_q \bullet K_p) = pq,$$

for $p, q \geq 3$

Proof. Let $V(K_q) = \{ t_w : 1 \leq w \leq q \}$ and $V(K_p) = \{ t_w' : 1 \leq w' \leq p \}$. Let $V(K_q \bullet K_p) = \{ t_w : 1 \leq w \leq q \} \cup \{ t_w' : 1 \leq w \leq q, 1 \leq w' \leq p \}$ be the node set of $K_q \bullet K_p$.

From the definition of Extended Corona, the graph $K_q \bullet K_p$ is obtained by taking corona of $K_q$ with $K_p$ and joining each vertex of $i$th copy of $K_p$ to every vertex of $j$th copy of $K_p$ provided that $t_i$ and $t_j$ are adjacent in $K_q$.

For $1 \leq w \leq q$, grant the color $w$ to the vertices $t_w$ in $K_q$. Now for $q$-copies of $K_p$, and for $2 \leq w \leq q$, grant the color as $w - 1$ to $t_w$ and when $w = 1$, the color $q$ to $t'_1$. And for $1 \leq w \leq q$, we have the color $(w - 1)q + w$ to $t'_w$, where $2 \leq w' \leq p$.

Each color 1, 2, 3, ..., $q$ cropped up twice, all the remaining color $q + 1, q + 2, ..., pq$ cropped once each respectively, and the absolute difference between the color classes is at most 1. The resultant graph $K_q \bullet K_p$ is equitably colored. The upper bound,

$$\chi_e = (K_q \bullet K_p) \leq pq,$$

for $p, q \geq 3$.

As above, we know that $\chi(K_q) \geq \chi(K)$ and $\chi(K_q \bullet K_p) = pq$. And we have $\chi_e = (K_q \bullet K_p) \geq \chi(K_q \bullet K_p) = pq$. The lower bound becomes,

$$\chi_e = (K_q \bullet K_p) \geq pq,$$

for $p, q \geq 3$.

Therefore, the extended corona product of $K_q$ with $K_p$ graph is

$$\chi_e = (K_q \bullet K_p) = pq,$$

for $p, q \geq 3$.  \[\square\]
In $\chi_-(K_q \bullet K_p)$, we have $\theta(c_i) = 2$ for $1 \leq i \leq q$ and $\theta(c_i) = 1$ for $q + 1 \leq i \leq pq$.

**Parameters of $\chi_-(K_q \bullet K_p)$**

- **Probability Mass Function**
  
  $f(i) = \begin{cases} 
  \frac{2}{q(p+1)}; & \text{if } 1 \leq i \leq q \\
  \frac{1}{q(p+1)}; & \text{if } q + 1 \leq i \leq pq \\
  0; & \text{otherwise} 
  \end{cases}$

- **Coloring Mean**
  
  $\mu_{\chi_-(K_q \bullet K_p)} = \begin{cases} 
  \frac{q+1}{2}; & \text{if } 1 \leq i \leq q \\
  \frac{qp + q + 1}{2}; & \text{if } q + 1 \leq i \leq pq 
  \end{cases}$

- **Coloring Variance**
  
  $\sigma^2_{\chi_-(K_q \bullet K_p)} = \begin{cases} 
  \frac{q^2 - 1}{2(p^2 q^2 + 3pq^2 - p - q^2 + 1)}; & \text{if } 1 \leq i \leq q \\
  \frac{12}{12(p-1)}; & \text{if } q + 1 \leq i \leq pq 
  \end{cases}$

**Theorem 3.2.** The equitable coloring of extended corona of complete graph $K_q$ with path graph $P_s$ is given by $\chi_-(K_q \bullet P_s) = 2q$, for $q \geq 3, s \geq 2$.

**Proof.** Let $V(K_q) = \{t_w : 1 \leq w \leq q\}$ and $V(P_s) = \{h_v : 1 \leq v \leq s\}$. Let $V(K_q \bullet P_s) = \{t_w : 1 \leq w \leq q\} \cup \{h_{vw} : 1 \leq w \leq q, 1 \leq v \leq s\}$ be the node set of $K_q \bullet P_s$.

From the definition of Extended Corona, the graph $K_q \bullet P_s$ is obtained by taking corona of $K_q$ with $P_s$ and joining each vertex of $i^{th}$ copy of $P_s$ to every vertex of $j^{th}$ copy of $P_s$ provided that $t_i$ and $t_j$ are adjacent in $K_q$.

For $1 \leq w \leq q$, we grant the color $w$ to the vertices $t_w$ in $K_q$. Now for $q$-copies of $P_s$ and for $2 \leq w \leq q$, grant the color as $w - 1$ to $h_{w(2v)}$ where $1 \leq v \leq \left\lfloor \frac{q}{2} \right\rfloor$. When $w = 1$ we have the color $q$ to $h_{w(2v)}$ $1 \leq v \leq \left\lfloor \frac{q}{2} \right\rfloor$. For $1 \leq w \leq q$, grant the color as $q + w$ to $h_{w(2v-1)}$ where $1 \leq v \leq \left\lfloor \frac{q}{2} \right\rfloor$.

If $s$ is even, each color $1, 2, ..., q$ will crop up $\left\lfloor \frac{q}{2} + 1 \right\rfloor$ and the color $q + 1, q + 2, ..., 2q$ will crop up $\frac{q}{2}$ respectively.

If $s$ is odd, each color $1, 2, ..., 2q$ will crop up $\left\lfloor \frac{q}{2} \right\rfloor$ respectively.

The absolute difference between the color classes is at most one. The resultant graph $K_q \bullet P_s$ is equitably colored. The upper bound, $\chi_-(K_q \bullet P_s) \leq 2q$, for $s \geq 2, q \geq 3$.

As above, we know that $\chi_-(K) \geq \chi(K)$ and $\chi(K_q \bullet P_s) = 2q$, and we have $\chi_-(K_q \bullet P_s) = 2q$. The lower bound becomes, $\chi_-(K_q \bullet P_s) \geq 2q$, for $s \geq 2, q \geq 3$.

Therefore, the extended corona product of $K_q$ with $P_s$ is $\chi_-(K_q \bullet P_s) = 2q$, for $s \geq 2, q \geq 3$. 

Parameters of $\chi_e(K_q \bullet P_s)$

- Probability Mass Function

(i) If $s$ is odd
\[
f(i) = \begin{cases} 
\frac{1}{2q}; & \text{if } 1 \leq i \leq 2q \\
0; & \text{otherwise}
\end{cases}
\]

(ii) If $s$ is even
\[
f(i) = \begin{cases} 
\frac{s + 2}{2q(s+1)}; & \text{if } 1 \leq i \leq q \\
\frac{s}{2q(s+1)}; & \text{if } q + 1 \leq i \leq 2q \\
0; & \text{otherwise}
\end{cases}
\]

- Coloring Mean

(i) If $s$ is odd
\[
\mu_{\chi_e}(K_q \bullet P_s) = \begin{cases} 
\frac{2q + 1}{2}; & \text{if } 1 \leq i \leq 2q \\
q + 1/2; & \text{if } q + 1 \leq i \leq 2q
\end{cases}
\]

(ii) If $s$ is even
\[
\mu_{\chi_e}(K_q \bullet P_s) = \begin{cases} 
\frac{q + 1}{2}; & \text{if } 1 \leq i \leq q \\
\frac{q + 1}{2}; & \text{if } q + 1 \leq i \leq 2q
\end{cases}
\]

- Coloring Variance

(i) If $s$ is odd
\[
\sigma^2_{\chi_e}(K_q \bullet P_s) = \begin{cases} 
\frac{4q^2 - 1}{12}; & \text{if } 1 \leq i \leq 2q \\
q^2/12; & \text{if } q + 1 \leq i \leq 2q
\end{cases}
\]

(ii) If $s$ is even
\[
\sigma^2_{\chi_e}(K_q \bullet P_s) = \begin{cases} 
\frac{q^2 - 1}{12}; & \text{if } 1 \leq i \leq q \\
\frac{q^2 - 1}{12}; & \text{if } q + 1 \leq i \leq 2q
\end{cases}
\]

Theorem 3.3. The equitable coloring of extended corona of path graph($P_s$) with complete graph($K_q$) is given by
\[
\chi_e(P_s \bullet K_q) = \begin{cases} 
2q + 1; & \text{if } s = 3, 4, 6; q \geq 3 \\
2(q + 1); & \text{if } s = 5 \text{ & } s \geq 7; q \geq 3
\end{cases}
\]

Proof. Let $V(P_s) = \{h_v : 1 \leq v \leq s\}$ and $V(K_q) = \{t_w : 1 \leq w \leq q\}$. Let $V(P_s \bullet K_q) = \{h_v : 1 \leq v \leq s\} \cup \{t_w : 1 \leq v \leq s, 1 \leq w \leq q\}$ be the node set of $P_s \bullet K_q$.

From the definition of Extended Corona, the graph $P_s \bullet K_q$ is obtained by taking corona of $P_s$ with $K_q$ and joining each vertex of $i^{th}$ copy of $K_q$ to every vertex of $j^{th}$ copy of $K_q$ provided that $h_i$ and $h_j$ are adjacent in $P_s$. 
(i) Case 1: If $q \geq 3, s = 3, 4, 6$

**Subcase 1:** When $q \geq 3, s = 3, 6$

For $1 \leq v \leq s$, we grant the color $v$ to the vertices $h_v$ in $P_s$. Now for $s$-copies of $K_q$, grant the color to $1$ to $t_{21}$ and $t_{13}$, $2$ to $t_{31}$ and $t_{16}$, $3$ to $t_{41}$ and $t_{141}$. And again the color $4$ to $t_{(2u-1)2}$, $5$ to $t_{(2u-1)3}$, $6$ to $t_{(2u-1)4}$, ..., $q + 2$ to $t_{(2u-1)q}$ where $1 \leq v \leq \left\lfloor \frac{q}{2} \right\rfloor$ respectively. Also the color $q + 3$ to $t_{2(2u)2}$, $q + 4$ to $t_{2(2u)3}$, ..., $2q + 1$ to $t_{2(2u)q}$ where $1 \leq v \leq \left\lfloor \frac{q}{2} \right\rfloor$ respectively.

**Subcase 2:** When $q \geq 3, s = 4$

For the graph $P_s \cdot K_q$, we have the color $1$ to $h_1$ & $h_4$, $2$ to $h_2$, $3$ to $h_3$ and $1$ to $t_{21}$, $2$ to $t_{31}$, $3$ to $t_{41}$ & $3$ to $t_{11}$ respectively.

Now the copies of complete graph $K_q$, for $1 \leq v \leq 2$ we have the color $w + 2$ to $t_{(2u-1)w}$ and the color $q + w + 1$ to $t_{(2e)w}$ where $2 \leq w \leq q$ respectively.

(ii) Case 2: If $q \geq 3, s = 5 & s \geq 7$

**Subcase 1:** When $s \equiv 0, 1, 3 \mod 4$

In $P_s \cdot K_q$, the nodes $h_v$ ($1 \leq v \leq s$) of $P_s$ have the color 1 to $h_v$ ($v \equiv 1 \mod 4$), 2 to $h_v$ ($v \equiv 2 \mod 4$), 3 to $h_v$ ($v \equiv 3 \mod 4$), 4 to $h_v$ ($v \equiv 0 \mod 4$) respectively. When $1 \leq v \leq s$, for the nodes $t_{ew}$ of $s$ copies of $K_q$ we grant the color 1 to $t_{e1}$ ($v \equiv 2 \mod 4$), 2 to $t_{e1}$ ($v \equiv 3 \mod 4$), 3 to $t_{e1}$ ($v \equiv 0 \mod 4$), 4 to $t_{e1}$ ($v \equiv 1 \mod 4$). For the remaining nodes 1 $\leq v \leq \left\lfloor \frac{q}{2} \right\rfloor$, we grant the color 5 to $t_{(2u-1)2}$, 6 to $t_{(2u-1)3}$, ..., $q + 3$ to $t_{(2u-1)q}$ respectively. The color $q + 4$ to $t_{(2e)2}$, $q + 5$ to $t_{(2e)3}$, ..., $2q + 1$ to $t_{(2e)q}$ where $1 \leq v \leq \left\lfloor \frac{q}{2} \right\rfloor$.

**Subcase 2:** When $s \equiv 2 \mod 4$

For $1 \leq v \leq s$ in $P_s$ we grant the color 1 to $h_v$ ($v \equiv 1 \mod 4$), 2 to $h_v$ ($v \equiv 2 \mod 4$), 3 to $h_v$ ($v \equiv 3 \mod 4$), 4 to $h_v$ ($v \equiv 0 \mod 4$) respectively. And when $1 \leq v \leq s - 1$, we grant the color 1 to $t_{e1}$ ($v \equiv 2 \mod 4$), 2 to $t_{e1}$ ($v \equiv 3 \mod 4$), 3 to $t_{e1}$ ($v \equiv 0 \mod 4$), 4 to $t_{e1}$ ($v \equiv 1 \mod 4$) and when $v = s$, the color 3 to $t_{e1}$ ($v \equiv 2 \mod 4$). Again for $1 \leq v \leq \left\lfloor \frac{q}{2} \right\rfloor$, we grant the color 5 to $t_{(2u-1)2}$, 6 to $t_{(2u-1)3}$, ..., $q + 3$ to $t_{(2u-1)q}$ and $q + 4$ to $t_{(2u)2}$, $q + 5$ to $t_{(2u)3}$, ..., $2q + 1$ to $t_{(2u)q}$.

The absolute difference between the color classes is at most 1. The resultant graph $P_s \cdot K_q$ is **equitably colored**. The upper bound,

$$\chi_{(P_s \cdot K_q)} \leq \begin{cases} 2q + 1; & \text{if } s = 3, 4, 6; q \geq 3 \\ 2(q + 1); & \text{if } s = 5 & s \geq 7; q \geq 3 \end{cases}$$

As above, we know that $\chi_{(K)} \geq \chi_{(K)}$ and $\chi_{(P_s \cdot K_q)} = 2q$. And we have $\chi_{(P_s \cdot K_q)} \geq \chi_{(P_s \cdot K_q)} = 2q$. The lower bound becomes,

$$\chi_{(P_s \cdot K_q)} \geq 2q, \text{ for } s \geq 2, q \geq 3.$$

Therefore, the extended corona product of $P_s$ with $K_q$ graph is

$$2q \leq \chi_{(P_s \cdot K_q)} \leq \begin{cases} 2q + 1; & \text{if } s = 3, 4, 6; q \geq 3 \\ 2(q + 1); & \text{if } s = 5 & s \geq 7; q \geq 3 \end{cases}$$

**Parameters of $\chi_{(P_s \cdot K_q)}$**

- **Probability Mass Function**
  For $s = 5 & s \geq 7, q \geq 3$
  (i) If $s \equiv 0, 2 \mod 4$

$$f(i) = \begin{cases} \frac{1}{2q + 2}; & \text{if } 1 \leq i \leq 2q + 2 \\ 0; & \text{otherwise} \end{cases}$$

(ii) If $s \equiv 1 \mod 4$

$$f(i) = \begin{cases} \frac{s + 1}{2s(q + 1)}; & \text{if } 1; 4 \leq i \leq q + 3 \\ \frac{s - 1}{2s(q + 1)}; & \text{if } 2, 3; q + 4 \leq i \leq 2(q + 1) \\ 0; & \text{otherwise} \end{cases}$$
(iii) If \( s \equiv 3 \mod 4 \)

\[
f(i) = \begin{cases} 
\frac{s + 1}{2s(q + 1)}; & \text{if } i = 1, 2; 5 \leq i \leq q + 3 \\
\frac{s - 1}{2s(q + 1)}; & \text{if } i = 3, 4; q + 4 \leq i \leq 2(q + 1) \\
0; & \text{otherwise}
\end{cases}
\]

• **Coloring Mean**

For \( s = 5 \) & \( s \geq 7, q \geq 3 \)

(i) If \( s \equiv 0, 2 \mod 4 \)

\[
\mu_{\chi_e}(P_s \bullet K_q) = \begin{cases} 
\frac{2q + 3}{2}; & \text{if } 1 \leq i \leq 2(q + 1)
\end{cases}
\]

(ii) If \( s \equiv 1 \mod 4 \)

\[
\mu_{\chi_e}(P_s \bullet K_q) = \begin{cases} 
\frac{q^2 + 7q + 2}{2(q + 1)}; & \text{if } i = 1; 4 \leq i \leq q + 3 \\
\frac{3q^2 + 3q + 4}{2(q + 1)}; & \text{if } i = 2, 3; q + 4 \leq i \leq 2(q + 1)
\end{cases}
\]

(iii) If \( s \equiv 3 \mod 4 \)

\[
\mu_{\chi_e}(P_s \bullet K_q) = \begin{cases} 
\frac{q^2 + 7q - 2}{2(q + 1)}; & \text{if } i = 1, 2; 5 \leq i \leq q + 3 \\
\frac{3q^2 + 3q + 8}{2(q + 1)}; & \text{if } i = 3, 4; q + 4 \leq i \leq 2(q + 1)
\end{cases}
\]

• **Coloring Variance**

For \( s = 5 \) & \( s \geq 7, q \geq 3 \)

(i) If \( s \equiv 0, 2 \mod 4 \)

\[
\sigma^2_{\chi_e}(P_s \bullet K_q) = \begin{cases} 
\frac{(2q + 3)(2q + 1)}{12}; & \text{if } 1 \leq i \leq 2(q + 1)
\end{cases}
\]

(ii) If \( s \equiv 1 \mod 4 \)

\[
\sigma^2_{\chi_e}(P_s \bullet K_q) = \begin{cases} 
\frac{q(q^3 + 4q^2 + 29q + 74)}{12(q + 1)^2}; & \text{if } i = 1; 4 \leq i \leq q + 3 \\
\frac{q^3 + 52q^2 - 19q - 22}{12(q + 1)^2}; & \text{if } i = 2, 3; q + 4 \leq i \leq 2(q + 1)
\end{cases}
\]

(iii) If \( s \equiv 3 \mod 4 \)

\[
\sigma^2_{\chi_e}(P_s \bullet K_q) = \begin{cases} 
\frac{q^4 + 4q^3 + 53q^2 + 98q - 144}{12(q + 1)^2}; & \text{if } i = 1, 2; 5 \leq i \leq q + 3 \\
\frac{q^3 + 52q^2 - 91q + 50}{12(q + 1)^2}; & \text{if } i = 3, 4; q + 4 \leq i \leq 2(q + 1)
\end{cases}
\]
Theorem 3.4. The equitable coloring of extended corona of complete graph \( K_q \) with cycle graph \( C_r \) graph for \( q, r \geq 3 \) is given by

\[
\chi_{\epsilon}(K_q \circ C_r) = \begin{cases} 
3q; & \text{if } r \text{ is odd} \\
2q; & \text{if } r \text{ is even}
\end{cases}
\]

Proof. Let \( V(K_q) = \{ t_w : 1 \leq w \leq q \} \) and \( V(C_r) = \{ g_u : 1 \leq u \leq r \} \). Let \( V(K_q \circ C_r) = \{ t_w : 1 \leq w \leq q \} \cup \{ g_{wu} : 1 \leq w \leq q, 1 \leq u \leq r \} \) be the node set of \( K_q \circ C_r \).

From the definition of Extended Corona, the graph \( K_q \circ C_r \) is obtained by taking corona of \( K_q \) with \( C_r \) and joining each vertex of \( i^{th} \) copy of \( C_r \) to every vertex of \( j^{th} \) copy of \( C_r \) provided that \( t_i \) and \( t_j \) are adjacent in \( K_q \).

(i) Case 1: If \( r \) is even

For \( 1 \leq w \leq q - 1 \), we grant the color \( 2w \) to the vertices \( t_{(w+1)} \) and the color \( 2q \) to \( t_1 \) in \( K_q \).

Now for \( q \)-copies of \( C_r \) and for \( 1 \leq u \leq \left \lceil \frac{w}{3} \right \rceil \), grant the color as \( 1 \) to \( g_{1(2u-1)} \), \( 2 \) to \( g_{1(2u)} \), \( 3 \) to \( g_{2(2u-1)} \), \( 4 \) to \( g_{2(2u)} \), \( 5 \) to \( g_{3(2u)} \), \( 6 \) to \( g_{3(2u-1)} \) and \( 7 \) to \( g_{3(2u-2)} \).

(ii) Case 2: If \( r \) is odd

For \( 1 \leq w \leq q - 1 \), we grant the color \( 3w \) to the vertices \( t_{(w+1)} \) and the color \( 3q \) to \( t_1 \) in \( K_q \).

Subcase 1: When \( r \equiv 0 \mod 3 \)

For \( 1 \leq u \leq r \), we grant the colors \( 1 \) to \( g_{1u} \), \( 2 \) to \( g_{1u} \), \( 3 \) to \( g_{1u} \), \( 4 \) to \( g_{2u} \), \( 5 \) to \( g_{2u} \), \( 6 \) to \( g_{2u} \), \( 7 \) to \( g_{2u} \) and \( 8 \) to \( g_{2u} \).

Subcase 2: When \( r \equiv 1 \mod 3 \)

For \( 1 \leq u \leq r - 1 \), we grant the colors \( 1 \) to \( g_{1u} \), \( 2 \) to \( g_{1u} \), \( 3 \) to \( g_{1u} \), \( 4 \) to \( g_{2u} \), \( 5 \) to \( g_{2u} \), \( 6 \) to \( g_{2u} \), \( 7 \) to \( g_{2u} \) and \( 8 \) to \( g_{2u} \).

Subcase 3: When \( r \equiv 2 \mod 3 \)

For \( 1 \leq u \leq r \), we grant the colors \( 1 \) to \( g_{1u} \), \( 2 \) to \( g_{1u} \), \( 3 \) to \( g_{1u} \), \( 4 \) to \( g_{2u} \), \( 5 \) to \( g_{2u} \), \( 6 \) to \( g_{2u} \), \( 7 \) to \( g_{2u} \) and \( 8 \) to \( g_{2u} \).

When \( r \) is even, every color 1, 3, 5, ..., \( 2q - 1 \) crop up \( \left( \frac{q}{2} \right) \) times and the color 2, 4, ..., \( 2q \) will crop up \( \left( \frac{q}{2} + 1 \right) \) times.

When \( r \) is odd

(i) \( r \equiv 0 \mod 3 \)

The color 1, 2, 4, 5, 7, 8, ..., \( 3q - 2 \), \( 3q - 1 \) will crop up \( \left( \frac{q}{2} \right) \) times each and the color 3, 6, 9, ..., \( 3q \) will crop up \( \left( \frac{q}{2} + 1 \right) \) times each.

(ii) \( r \equiv 1 \mod 3 \)

The color 1, 4, 7, 10, ..., \( 3q - 2 \) will crop up \( \left( \frac{q}{2} \right) \) times each and the color 2, 3, 5, 6, 9, ..., \( 3q - 1 \) will crop up \( \left( \frac{q}{2} - 1 \right) \) times each.

(iii) \( r \equiv 2 \mod 3 \)

The color 1, 2, 3, ..., \( 3q \) will crop up \( \left( \frac{q + 1}{2} \right) \) times each.

The absolute difference between the color classes is at most 1. The resultant graph \( K_q \circ C_r \) is equitably colored. The upper bound,

\[
\chi_{\epsilon}(K_q \circ C_r) \leq \begin{cases} 
3q; & \text{if } r \text{ is odd} \\
2q; & \text{if } r \text{ is even}
\end{cases}
\]

As above, we know that \( \chi(K) \geq \chi(K) \) and \( \chi(K_q \circ C_r) = 2q \). And we have \( \chi(K_q \circ C_r) \geq \chi(K_q \circ C_r) = 2q \). The lower bound becomes,

\[
\chi(K_q \circ C_r) \geq 2q, \text{ for } q, r \geq 3.
\]
Therefore, the extended corona product of $K_q$ with $C_r$ graph is

$$2q \leq \chi_e(K_q \bullet C_r) \leq \begin{cases} 3q; & \text{if } r \text{ is odd} \\ 2q; & \text{if } r \text{ is even} \end{cases}$$

**Parameters of $\chi_e(K_q \bullet C_r)$**

- **Probability Mass Function**
  
  (i) If $r$ is even
  
  $$f(i) = \begin{cases} \frac{r}{2q(r + 1)}; & \text{if } i \text{ is odd} \\ \frac{r + 2}{2q(r + 1)}; & \text{if } i \text{ is even} \\ 0; & \text{otherwise} \end{cases}$$

  (ii) If $r$ is odd

  (a) $r \equiv 0 \mod 3$
  
  $$f(i) = \begin{cases} \frac{r + 3}{3q(r + 1)}; & \text{if } i = 3, 6, 9, ..., 3q \\ \frac{r}{3q(r + 1)}; & \text{if } i = 1, 2, 4, 5, ..., 3q - 2, 3q - 1 \\ 0; & \text{otherwise} \end{cases}$$

  (b) $r \equiv 1 \mod 3$
  
  $$f(i) = \begin{cases} \frac{r + 2}{3q(r + 1)}; & \text{if } i = 2, 3, 5, 6, ..., 3q - 1, 3q \\ \frac{r - 1}{3q(r + 1)}; & \text{if } i = 1, 4, 7, 10, ..., 3q - 2 \\ 0; & \text{otherwise} \end{cases}$$

  (c) $r \equiv 2 \mod 3$
  
  $$f(i) = \begin{cases} \frac{1}{5q}; & \text{if } 1 \leq i \leq 3q \\ 0; & \text{otherwise} \end{cases}$$

- **Coloring Mean**
  
  (i) If $r$ is even
  
  $$\mu_{\chi_e}(K_q \bullet C_r) = \begin{cases} q; & \text{if } i \text{ is odd} \\ q + 1; & \text{if } i \text{ is even} \end{cases}$$

  (ii) If $s$ is odd

  (a) $r \equiv 0 \mod 3$
  
  $$\mu_{\chi_e}(K_q \bullet C_r) = \begin{cases} \frac{3q}{2}; & \text{if } i = 1, 2, 4, 5, ..., 3q - 2, 3q - 1 \\ \frac{3(q + 1)}{2}; & \text{if } i = 3, 6, 9, ..., 3q \end{cases}$$
(b) \( r \equiv 1 \mod 3 \)

\[
\mu_{\chi_e}(K_q \cdot C_r) = \begin{cases} 
\frac{3q-1}{2}; & \text{if } i = 1, 4, 7, 9, \ldots, 3q - 2 \\
\frac{3(q+2)}{2}; & \text{if } i = 2, 3, 5, 6, \ldots, 3q - 1, 3q
\end{cases}
\]

(c) \( r \equiv 2 \mod 3 \)

\[
\mu_{\chi_e}(K_q \cdot C_r) = \frac{3q+1}{2}; \text{ if } i = 1, 2, 3, \ldots, 3q
\]

- Coloring Variance

(i) If \( r \) is even

\[
\sigma^2_{\chi_e}(P_s \cdot K_q) = \begin{cases} 
\frac{q^2 - 1}{3}; & \text{if } i \text{ is odd} \\
\frac{q^2 - 1}{3}; & \text{if } i \text{ is even}
\end{cases}
\]

(i) If \( r \) is even

(a) If \( r \equiv 0 \mod 3 \)

\[
\sigma^2_{\chi_e}(K_q \cdot C_r) = \begin{cases} 
\frac{3q^2 - 2}{4}; & \text{if } i = 1, 2, 4, 5, \ldots, 3q - 2, 3q - 1 \\
\frac{3(q^2 - 1)}{4}; & \text{if } i = 3, 6, 9, \ldots, 3q
\end{cases}
\]

(b) If \( r \equiv 1 \mod 3 \)

\[
\sigma^2_{\chi_e}(K_q \cdot C_r) = \begin{cases} 
\frac{3(q^2 - 1)}{4}; & \text{if } i = 1, 4, 7, 10, \ldots, 3q - 2 \\
\frac{3q^2 - 2}{4}; & \text{if } i = 2, 3, 5, 6, \ldots, 3q - 1, 3q
\end{cases}
\]

(c) If \( r \equiv 2 \mod 3 \)

\[
\sigma^2_{\chi_e}(K_q \cdot C_r) = \frac{9q^2 - 1}{12}; \text{ if } 1 \leq i \leq 3q
\]

\textbf{Theorem 3.5.} The equitable coloring of extended corona of cycle graph \((C_r)\) with complete graph \((K_q)\) is given by

When \( r \) is even

\[
\chi_e(C_r \cdot K_q) = 2q, \text{ if } q, r \geq 3
\]

When \( r \) is odd

\[
\chi_e(C_r \cdot K_q) = \begin{cases} 
3q; & \text{if } 3 \leq r \leq r - \left\lfloor \frac{r}{3} \right\rfloor \\
3q + 1; & \text{if otherwise}
\end{cases} \text{ if } r \equiv 0, 1 \mod 3
\]

\[
\begin{cases} 
3q; & \text{if } r \equiv 2 \mod 3
\end{cases}
\]

\[
\begin{cases} 
3q; & \text{if } 3 \leq r \leq r - \left\lfloor \frac{r}{3} \right\rfloor \\
3q + 1; & \text{if otherwise}
\end{cases} \text{ if } r \equiv 2 \mod 3
\]
Proof. Let \( V(C_r) = \{g_u : 1 \leq u \leq r\} \) and \( V(K_q) = \{t_w : 1 \leq w \leq q\} \). Let \( V(C_r \ast K_q) = \{g_u : 1 \leq u \leq r\} \cup \{t_{uw} : 1 \leq u \leq r, 1 \leq w \leq q\} \) be the node set of \( C_r \ast K_q \).

From the definition of Extended Corona, the graph \( C_r \ast K_q \) is obtained by taking corona of \( C_r \) with \( K_q \) and joining each vertex of \( i^{th} \) copy of \( K_q \) to every vertex of \( j^{th} \) copy of \( K_q \). So, when coloring the nodes of \( C_r \ast K_q \), provided that \( g_i \) and \( g_j \) are adjacent in \( C_r \).

(i) Case 1: If \( r \) is even
For \( 1 \leq u \leq \frac{r}{2} \), we grant the color \( w \) to \( t_{(2u-1)w} \) and the color \( q + w \) to \( t_{(2u)w} \) respectively where \( 1 \leq w \leq q \).

Subcase 1: When \( q \geq \frac{r}{2} \)
For \( 1 \leq u \leq \frac{r}{2} \) the nodes of \( C_r \), we grant the color \( u \) to \( g_{(2u)} \) and \( q + u \) to \( g_{2u-1} \).

Subcase 2: When \( q \leq \frac{r}{2} \)
For \( 1 \leq u \leq \frac{r}{2} \), we grant the color 1, 2,...,\( q \) repeatedly to the nodes \( g_{2u} \) and \( q + 1, ..., 2q \) to the nodes \( g_{2u-1} \) respectively.

(ii) Case 2: If \( r \) is odd
Subcase 1: When \( r \equiv 0 \mod 3 \)
For \( 1 \leq u \leq r \). If \( u \equiv 1 \mod 3 \) we grant the color \( w \) to \( t_{uw} \), if \( u \equiv 2 \mod 3 \) we grant the color \( q + w \) to \( t_{uw} \), if \( u \equiv 0 \mod 3 \) we grant the color \( 2q + w \) to \( t_{uw} \) where \( 1 \leq w \leq q \).

For the graph \( C_r \) and \( 1 \leq u \leq \frac{r}{2} \), we grant the colors 1, 2,...,\( q \) repeatedly to \( g_u \) (\( u \equiv 2 \mod 3 \)), the colors \( q + 1, q + 2, ..., 2q \) repeatedly to \( g_u \) (\( u \equiv 0 \mod 3 \)), \( 2q + 1, 2q + 2, ..., 3q \) repeatedly to \( g_u \) (\( u \equiv 1 \mod 3 \)).

Subcase 2: When \( r \equiv 1 \mod 3 \)
For \( 1 \leq u \leq r - 1 \). If \( u \equiv 1 \mod 3 \), we grant the colors \( w \) to \( t_{uw} \). If \( u \equiv 2 \mod 3 \), we grant the colors \( q + w \) to \( t_{uw} \), if \( u \equiv 0 \mod 3 \), we grant the colors \( 2q + w \) to \( t_{uw} \) where \( 1 \leq w \leq q \). When \( u = r \), we grant the color \( q + w \) to \( t_{uw} \) where \( 1 \leq w \leq q \).

In the subcase 2, each of the colors 1, 2,...,\( q \) have been occurred \( \frac{r-1}{3} \) times and the colors \( q + 1, ..., 2q \) have been occurred \( \frac{r+2}{3} \) in the copies of \( K_q \). While the left over colors \( 2q + 1, 2q + 2, ..., 3q \) has occurred \( \frac{r-1}{3} \) times in the copies of \( K_q \). So, when coloring the nodes of \( C_r \), one must use the colors 1, 2,...,\( 2q \) first for \( g_u \) where \( 1 \leq u \leq r \) and without altering the equitable coloring condition.

Subcase 3: When \( r \equiv 2 \mod 3 \)
(a) If \( 3 \leq q \leq r - \left\lfloor \frac{r}{2} \right\rfloor \)
For \( 1 \leq u \leq r \) and \( u \equiv 1 \mod 3 \), we grant the color \( w \) to \( t_{uw} \), \( u \equiv 2 \mod 3 \) we grant the color \( q + w \) to \( t_{uw} \) and \( u \equiv 0 \mod 3 \) we grant the color \( 2q + w \) to \( t_{uw} \) where \( 1 \leq w \leq q \).

By above, each of the colors 1, 2,...,\( 2q \) have been occurred \( \frac{r+1}{3} \) times in the copies of \( K_q \). While the left over colors \( 2q + 1, 2q + 2, ..., 3q \) has occurred \( \frac{r-2}{3} \) times each in copies of \( K_q \). So, when coloring the nodes of \( C_r \), one must use the colors \( 2q + 1, 2q + 2, ..., 3q \) first for \( g_u \) where \( 1 \leq u \leq r \) and without altering the equitable coloring condition.

(b) Otherwise
For \( 1 \leq u \leq r \) and \( u \equiv 1 \mod 3 \), we grant the color \( w \) to \( t_{uw} \) and \( u \equiv 2 \mod 3 \) we grant the color \( q + w \) to \( t_{uw} \) and \( u \equiv 0 \mod 3 \) we grant the color \( 2q + w \) to \( t_{uw} \) where \( 1 \leq w \leq q \).

For \( 1 \leq u \leq r \). We grant the color \( 2q + 1, 2q + 2, ..., 3q \) to the nodes \( g_u \) where \( u \equiv 1, 2 \mod 3 \). When \( u \equiv 0 \mod 3 \), we have the color \( 3q + 1 \).

The absolute difference between the color classes is at most 1. The resultant graph \( C_r \ast K_q \) is **equitably colored**. The upper bound, When \( r \) is even

\[
\chi(C_r \ast K_q) \leq 2q, \text{ if } q \geq 3
\]
When $r$ is odd

$$\chi=(C_r \bullet K_q) \leq \begin{cases} 3q; & \text{if } r \equiv 0,1 \mod 3 \\ 3q; & \text{if } 3 \leq r \leq r - \left\lfloor \frac{r}{3} \right\rfloor \\ 3q + 1; & \text{if } \text{otherwise} \end{cases}$$

As above, we know that $\chi(K) \geq \chi(K)$ and $\chi(C_r \bullet K_q) = 2q$. And we have $\chi=(C_r \bullet K_q) \geq \chi(C_r \bullet K_q) = 2q$. The lower bound becomes,

$$\chi=(C_r \bullet K_q) \geq 2q, \text{ for } s \geq 2, q \geq 3.$$

Therefore, the extended corona product of $C_r$ with $K_q$ graph is

When $r$ is even

$$\chi=(C_r \bullet K_q) = 2q, \text{ if } q, r \geq 3$$

When $r$ is odd

$$2q \leq \chi=(C_r \bullet K_q) \leq \begin{cases} 3q; & \text{if } r \equiv 0,1 \mod 3 \\ 3q; & \text{if } 3 \leq r \leq r - \left\lfloor \frac{r}{3} \right\rfloor \\ 3q + 1; & \text{if } \text{otherwise} \end{cases}$$

Conclusion

Finding p.m.f, mean, variance of equitable coloring of $C_r \bullet K_q$ has so many cases in it. It’s a tedious process for generalizing the occurrence of each color because in some case, we colored randomly for some vertices. Hence, this part alone is an open problem to the reader.

References


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