Palestine Journal of Mathematics

Vol. 10 (Special Issue II, 2021), 131–142

ON EQUITABLE COLORING OF EXTENTED CORONA OF SOME GRAPHS

M. Barani, M.Venkatachalam and K.Rajalakshmi

Communicated by E. Prakash

MSC 2010 Classifications: 05C15.

Keywords and phrases: Equitable coloring, Extented Corona, Cycle graph, Path graph and Complete graph.

Abstract A proper $h-$colorable graph K is claimed to be $equitably\ h-colorable$ if the vertex set of K can be partioned into h independent color classes $V_1, V_2, ..., V_h$ such that the condition $\left||V_{i'}| - \left|V_{j'}\right|\right| \leq 1$ holds for all different pairs of $(i^{'}, j^{'})$. And the smallest integer h is called $equitable\ chromatic\ number$ of K [5, 7]. In this paper, we consider an equitable coloring of extended corona product of two graphs K and H. In particular, we study the cases where K and H are complete graphs, cycles and paths. We also discussed the probability mass function, chromatic mean, and chromatic variance of these graph products.

1 Introduction

Many real world situations can coveniently be described by means of a diagram consisting of set of points together with lines joining certain pairs of these points [1].

The syllabary of an equitable coloring was first initiated by W. Meyer in the year 1973 [8]. Tucker's paper [14], in which nodes represent garbage collection routes and adjacency of two such vertices when the corresponding routes should not be run on the same day. Then Meyer come up with the solution that equal number of routes run on every day in a week.

The graph K is *equitably colored* with h colors, if the absolute differnce between thier color classes are atmost one[6, 9]. The smallest integer h for which K is equitably h-colorable is known as the *equitable chromatic number* of K and denoted by $\chi_=(K)$. Since an equitable coloring is a proper vertex coloring, we have

$$\chi_=(K) \geq \chi(K),$$

where $\chi(K)$ is the chromatic number of graph K.

2 Preliminaries

The corona $K_1 \circ K_2$ of two graphs K_1 and K_2 (where K_i has p_i points and q_i lines) is defined as the graph K obtained by taking one copy of K_1 and p_i copies of K_2, and then joining by a line the $i^{'}$th point of K_1 to every point in the $i^{'}$th copy of K_2 [4].

The extended corona $K_1 \bullet K_2$ of two graphs K_1 and K_2 is a graph obtained by taking the corona $K_1 \circ K_2$ and joining each vertex of $i^{'}$th copy of K_2 to every vertex of $j^{'}$th copy of K_2, provided the vertices $v_{i'}$ and $v_{j'}$ are adjacent in K_1 [2].

A $path$ is a non-empty graph $P = (V(P), E(P))$ of the form $V(P) = \{a_1, a_2, ..., a_s\}$ and $E(P) = \{a_1a_2, a_2a_3, ..., a_{s-1}a_s\}$. The number vertices of a path is its $length$ [1, 3].

A $cycle$ is a closed trail $r \geq 3$, for which each vertices are distict(expect, of course first and last) [3].

A $complete$ graph K_q has q points and all two-element subsets of K_q as edges [3, 10].

Let $\{c_1, c_2, c_3, \ldots, c_h\}$ be a set of colors used in a proper h-coloring $\mathcal{C}$ of K and let $\theta(c_i)$ denote the number of times a particular color c_i is assigned to nodes of K [6]. Let X be a random variable $(r.v)$ which denotes the color of an arbitrarily chosen vertex in K. Since the sum of all weights of colors of K is the order of K, the real valued function $f(i)$ defined by

$$f(i) = \begin{cases} \dfrac{\theta(c_i)}{|V(K)|} & i = 1, 2, 3, ..., h; \\ 0; & \text{elsewhere} \end{cases}$$

is the **probability mass function(p.m.f)** of the X [11, 12, 13]. Here we say $f(i)$ is the $p.m.f$ of the graph K with respect to the given coloring $\mathcal{C}$.

The **coloring mean** [11, 12, 13] of a coloring $\mathcal{C}$ of a given graph K is denoted by $\mu_c(K)$ and defined to be

$$\mu_c(K) = \frac{\sum\limits_{i=1}^{h} i\theta(c_i)}{\sum\limits_{i=1}^{h} \theta(c_i)}$$

The **coloring variance** [11, 12, 13] of a coloring $\mathcal{C}$ of a given graph K is denoted by $\sigma_c^2(K)$ and defined to be

$$\sigma_c^2(K) = \frac{\sum\limits_{i=1}^{h} i^2\theta(c_i)}{\sum\limits_{i=1}^{h} \theta(c_i)} - \left(\frac{\sum\limits_{i=1}^{h} i\theta(c_i)}{\sum\limits_{i=1}^{h} \theta(c_i)} \right)^2$$

The chromatic mean and variance corresponding to an equitable coloring of a graph K is defined as follows [12]. A coloring mean of a graph K, with respect to a proper coloring $\mathcal{C}$ is said to be an **equitable chromatic mean** or a χ_e chromatic mean of K, if $\mathcal{C}$ is the optimal (in the sense of the number of colors) equitable coloring of K. The χ_e-chromatic mean of a graph K is denoted by $\mu_{\chi_e}(K)$.

3 Results

Theorem 3.1. *The equitable coloring of extended corona of complete graph(K_q) with complete graph(K_p) is given by*

$$\chi_=(K_q \bullet K_p) = pq, \quad for\ p, q \geq 3$$

Proof. Let $V(K_q) = \{t_w : 1 \leq w \leq q\}$ and $V(K_p)$=$\{t'_{w'} : 1 \leq w' \leq p\}$. Let $V(K_q \bullet K_p) = \{t_w : 1 \leq w \leq q\} \bigcup \{t'_{ww'} : 1 \leq w \leq q, 1 \leq w' \leq p\}$ be the node set of $K_q \bullet K_p$.

From the definition of Extended Corona, the graph $K_q \bullet K_p$ is obtained by taking corona of K_q with K_p and joining each vertex of i^{th} copy of K_p to every vertex of j^{th} copy of K_p provided that t_i and t_j are adjacent in K_q.

For $1 \leq w \leq q$, grant the color w to the vertices t_w in K_q. Now for q-copies of K_p and for $2 \leq w \leq q$, grant the color as $w - 1$ to t'_{w1} and when $w = 1$, the color q to t'_{w1}. And for $1 \leq w \leq q$, we have the color $(w' - 1)q + w$ to $t'_{ww'}$ where $2 \leq w' \leq p$.

Each color $1, 2, 3, ..., q$ cropped up twice, all the remaining color $q + 1, q + 2, ..., pq$ cropped once each respectively, and the absolute difference between the color classes is at most 1. The resultant graph $K_q \bullet K_p$ is *equitably colored*. The upper bound,

$$\chi_=(K_q \bullet K_p) \leq pq, \quad \text{for } p, q \geq 3.$$

As above, we know that $\chi_=(K) \geq \chi(K)$ and $\chi(K_q \bullet K_p) = pq$. And we have $\chi_=(K_q \bullet K_p) \geq \chi(K_q \bullet K_p) = pq$. The lower bound becomes,

$$\chi_=(K_q \bullet K_p) \geq pq, \quad \text{for } p, q \geq 3.$$

Therefore, the extended corona product of K_q with K_p graph is

$$\chi_=(K_q \bullet K_p) = pq, \quad \text{for } p, q \geq 3.$$

□

In $\chi_{=}(K_q \bullet K_p)$, we have $\theta(c_i) = 2$ for $1 \leq i \leq q$ and $\theta(c_i) = 1$ for $q+1 \leq i \leq pq$.

Parameters of $\chi_{=}(K_q \bullet K_p)$

- *Probability Mass Function*

$$f(i) = \begin{cases} \dfrac{2}{q(p+1)}; & \text{if } \; 1 \leq i \leq q \\ \dfrac{1}{q(p+1)}; & \text{if } \; q+1 \leq i \leq pq \\ 0; & \text{otherwise} \end{cases}$$

- *Coloring Mean*

$$\mu_{\chi_e}(K_q \bullet K_p) = \begin{cases} \dfrac{q+1}{2}; & \text{if } \; 1 \leq i \leq q \\ \dfrac{qp+q+1}{2}; & \text{if } \; q+1 \leq i \leq pq \end{cases}$$

- *Coloring Variance*

$$\sigma^2_{\chi_e}(K_q \bullet K_p) = \begin{cases} \dfrac{q^2-1}{12}; & \text{if } \; 1 \leq i \leq q \\ \dfrac{p^3q^2 - 3p^2q^2 + 3pq^2 - p - q^2 + 1}{12(p-1)}; & \text{if } \; q+1 \leq i \leq pq \end{cases}$$

Theorem 3.2. *The equitable coloring of extended corona of complete graph(K_q) with path graph(P_s) is given by*

$$\chi_{=}(K_q \bullet P_s) = 2q, \quad for \; q \geq 3, s \geq 2$$

Proof. Let $V(K_q) = \{t_w : 1 \leq w \leq q\}$ and $V(P_s)$=$\{h_v : 1 \leq v \leq s\}$. Let $V(K_q \bullet P_s) = \{t_w : 1 \leq w \leq q\} \bigcup \{t'_{wv} : 1 \leq w \leq q, 1 \leq v \leq s\}$ be the node set of $K_q \bullet P_s$.

From the definition of Extended Corona, the graph $K_q \bullet P_s$ is obtained by taking corona of K_q with P_s and joining each vertex of i^{th} copy of P_s to every vertex of j^{th} copy of P_s provided that t_i and t_j are adjacent in K_q.

For $1 \leq w \leq q$, we grant the color w to the vertices t_w in K_q. Now for q-copies of P_s and for $2 \leq w \leq q$, grant the color as $w-1$ to $h_{w(2v)}$ where $1 \leq v \leq \lfloor \frac{s}{2} \rfloor$. When $w = 1$ we have the color q to $h_{w(2v)}$ $1 \leq v \leq \lfloor \frac{s}{2} \rfloor$. For $1 \leq w \leq q$, grant the color as $q+w$ to $h_{w(2v-1)}$ where $1 \leq v \leq \lceil \frac{s}{2} \rceil$.

If s is $even$, each color $1, 2, ..., q$ will crop up $\left(\frac{s}{2}+1\right)$ and the color $q+1, q+2, ..., 2q$ will crop up $\frac{s}{2}$ respectively.

If s is odd, each color $1, 2, ..., 2q$ will crop up $\lceil \frac{s}{2} \rceil$ respectively.

The absolute difference between the color classes is at most one. The resultant graph $K_q \bullet P_s$ is $equitably\ colored$. The upper bound,

$$\chi_{=}(K_q \bullet P_s) \leq 2q, \quad \text{for } s \geq 2, q \geq 3$$

As above, we know that $\chi_{=}(K) \geq \chi(K)$ and $\chi(K_q \bullet P_s) = 2q$, and we have $\chi_{=}(K_q \bullet P_s) \geq \chi(K_q \bullet P_s) = 2q$. The lower bound becomes,

$$\chi_{=}(K_q \bullet P_s) \geq 2q, \quad \text{for } s \geq 2, q \geq 3.$$

Therefore, the extended corona product of K_q with P_s is

$$\chi_{=}(K_q \bullet P_s) = 2q, \quad \text{for } s \geq 2, q \geq 3.$$

□

Parameters of $\chi_{=}(K_q \bullet P_s)$

- *Probability Mass Function*

(i) If s is odd

$$f(i) = \begin{cases} \frac{1}{2q}; & \textit{if} \quad 1 \leq i \leq 2q \\ 0; & \text{otherwise} \end{cases}$$

(ii) If s is $even$

$$f(i) = \begin{cases} \frac{s+2}{2q(s+1)}; & \textit{if} \quad 1 \leq i \leq q \\ \frac{s}{2q(s+1)}; & \textit{if} \quad q+1 \leq i \leq 2q \\ 0; & \text{otherwise} \end{cases}$$

- *Coloring Mean*

(i) If s is odd

$$\mu_{\chi_e}(K_q \bullet P_s) = \begin{cases} \frac{2q+1}{2}; & \textit{if} \quad 1 \leq i \leq 2q \end{cases}$$

(ii) If s is $even$

$$\mu_{\chi_e}(K_q \bullet P_s) = \begin{cases} \frac{q+1}{2}; & \textit{if} \quad 1 \leq i \leq q \\ \frac{3q+1}{2}; & \textit{if} \quad q+1 \leq i \leq 2q \end{cases}$$

- *Coloring Variance*

(i) If s is odd

$$\sigma^2_{\chi_e}(K_q \bullet P_s) = \begin{cases} \frac{4q^2-1}{12}; & \textit{if} \quad 1 \leq i \leq 2q \end{cases}$$

(ii) If s is $even$

$$\sigma^2_{\chi_e}(K_q \bullet P_s) = \begin{cases} \frac{q^2-1}{12}; & \textit{if} \quad 1 \leq i \leq q \\ \frac{q^2-1}{12}; & \textit{if} \quad q+1 \leq i \leq 2q \end{cases}$$

Theorem 3.3. *The equitable coloring of extended corona of path graph(P_s) with complete graph(K_q) is given by*

$$\chi_{=}(P_s \bullet K_q) = \begin{cases} 2q+1; & \textit{if} \quad s = 3,4,6; q \geq 3 \\ 2(q+1); & \textit{if} \quad s = 5 \ \& \ s \geq 7; q \geq 3 \end{cases}$$

Proof. Let $V(P_s)$=$\{h_v : 1 \leq v \leq s\}$ and $V(K_q) = \{t_w : 1 \leq w \leq q\}$. Let $V(P_s \bullet K_q) = \{h_v : 1 \leq v \leq s\} \bigcup \{\text{t}_{vw} : 1 \leq v \leq s, 1 \leq w \leq q\}$ be the node set of $P_s \bullet K_q$.

From the definition of Extended Corona, the graph $P_s \bullet K_q$ is obtained by taking corona of P_s with K_q and joining each vertex of i^{th} copy of K_q to every vertex of j^{th} copy of K_q provided that h_i and h_j are adjacent in P_s.

(i) **Case 1**: If $q \geq 3$, $s = 3, 4, 6$

Subcase 1: When $q \geq 3$, $s = 3, 6$

For $1 \leq v \leq s$, we grant the color v to the vertices h_v in P_s. Now for s-copies of K_q, grant the color to 1 to t_{21} and t_{51}, 2 to t_{31} and t_{61}, 3 to t_{41} and t_{11}. And again the color 4 to $t_{(2v-1)2}$, 5 to $t_{(2v-1)3}$, 6 to $t_{(2v-1)4}$,...,$q+2$ to $t_{(2v-1)q}$ where $1 \leq v \leq \lceil \frac{s}{2} \rceil$ respectively. Also the color $q+3$ to $t_{(2v)2}$, $q+4$ to $t_{(2v)3}$,...,$2q+1$ to $t_{(2v)q}$ where $1 \leq v \leq \lfloor \frac{s}{2} \rfloor$respectively.

Subcase 2: When $q \geq 3$, $s = 4$

For the graph $P_s \bullet K_q$, we have the color 1 to h_1 & h_4, 2 to h_2, 3 to h_3 and 1 to t_{21}, 2 to t_{31}, 3 to t_{41} & 3 to t_{11} respectively.

Now the copies of complete graph K_q, for $1 \leq v \leq 2$ we have the color $w+2$ to $t_{(2v-1)w}$ and the color $q+w+1$ to $t_{(2v)w}$ where $2 \leq w \leq q$ respectively.

(ii) **Case 2**: If $q \geq 3$, $s = 5$ & $s \geq 7$

Subcase 1: When $s \equiv 0, 1, 3 \mod 4$

In $P_s \bullet K_q$, the nodes h_v $(1 \leq v \leq s)$ of P_s have the color 1 to h_v $(v \equiv 1 \mod 4)$, 2 to h_v $(v \equiv 2 \mod 4)$, 3 to h_v $(v \equiv 3 \mod 4)$, 4 to h_v $(v \equiv 0 \mod 4)$ respectively. When $1 \leq v \leq s$, for the nodes t_{vw} of s copies of K_q we grant the color 1 to t_{v1} $(v \equiv 2 \mod 4)$, 2 to t_{v1} $(v \equiv 3 \mod 4)$, 3 to t_{v1} $(v \equiv 0 \mod 4)$, 4 to t_{v1} $(v \equiv 1 \mod 4)$. For the remaining nodes $1 \leq v \leq \lceil \frac{s}{2} \rceil$, we grant the color 5 to $t_{(2v-1)2}$, 6 to $t_{(2v-1)3}$, 7 to $t_{(2v-1)4}$,..., $q+3$ to $t_{(2v-1)q}$ respectiely. The color $q+4$ to $t_{(2v)2}$, $q+5$ to $t_{(2v)3}$,..., $2(q+1)$ to $t_{(2v)q}$ where $1 \leq v \leq \lfloor \frac{s}{2} \rfloor$.

Subcase 2: When $s \equiv 2 \mod 4$

For $1 \leq v \leq s$ in P_s we grant the color 1 to h_v $(v \equiv 1 \mod 4)$, 2 to h_v $(v \equiv 2 \mod 4)$, 3 to h_v $(v \equiv 3 \mod 4)$, 4 to h_v $(v \equiv 0 \mod 4)$ respectively. And when $1 \leq v \leq s-1$, we grant the color 1 to t_{v1} $(v \equiv 2 \mod 4)$, 2 to t_{v1} $(v \equiv 3 \mod 4)$, 3 to t_{v1} $(v \equiv 0 \mod 4)$, 4 to t_{v1} $(v \equiv 1 \mod 4)$ and when $v = s$, the color 3 to t_{v1} $(v \equiv 2 \mod 4)$. Again for $1 \leq v \leq \lceil \frac{s}{2} \rceil$, we grant the 5 to $t_{(2v-1)2}$, 6 to $t_{(2v-1)3}$,..., $q+3$ to $t_{(2v-1)q}$ and $q+4$ to $t_{(2v)2}$, $q+5$ to $t_{(2v)3}$,..., $2(q+1)$ to $t_{(2v)q}$.

The absolute difference between the color classes is at most 1. The resultant graph $P_s \bullet K_q$ is $equitably\ colored$. The upper bound,

$$\chi_{=}(P_s \bullet K_q) \leq \begin{cases} 2q+1; & \textit{if} \quad s = 3,4,6; q \geq 3 \\ 2(q+1); & \textit{if} \quad s = 5 \ \& \ s \geq 7; q \geq 3 \end{cases}$$

As above, we know that $\chi_{=}(K) \geq \chi(K)$ and $\chi(P_s \bullet K_q) = 2q$. And we have $\chi_{=}(P_s \bullet K_q \geq \chi(P_s \bullet K_q) = 2q$. The lower bound becomes,

$$\chi_{=}(P_s \bullet K_q) \geq 2q, \quad \text{for } s \geq 2, q \geq 3.$$

Therefore, the extended corona product of P_s with K_q graph is

$$2q \leq \chi_{=}(P_s \bullet K_q) \leq \begin{cases} 2q+1; & \textit{if} \quad s = 3,4,6; q \geq 3 \\ 2(q+1); & \textit{if} \quad s = 5 \ \& \ s \geq 7; q \geq 3 \end{cases}$$ □

Parameters of $\chi_{=}(P_s \bullet K_q)$

- *Probability Mass Function*
  For $s = 5$ & $s \geq 7, q \geq 3$
  (i) If $s \equiv 0, 2 \mod 4$

$$f(i) = \begin{cases} \dfrac{1}{2q+2}; & \textit{if} \quad 1 \leq i \leq 2q+2 \\ 0; & \text{otherwise} \end{cases}$$

  (ii) If $s \equiv 1 \mod 4$

$$f(i) = \begin{cases} \dfrac{s+1}{2s(q+1)}; & \textit{if} \quad i = 1; 4 \leq i \leq q+3 \\ \dfrac{s-1}{2s(q+1)}; & \textit{if} \quad i = 2,3; q+4 \leq i \leq 2(q+1) \\ 0; & \text{otherwise} \end{cases}$$

(iii) If $s \equiv 3 \mod 4$

$$f(i) = \begin{cases} \dfrac{s+1}{2s(q+1)}; & \textit{if} \quad i = 1, 2; 5 \leq i \leq q+3 \\ \dfrac{s-1}{2s(q+1)}; & \textit{if} \quad i = 3, 4; q+4 \leq i \leq 2(q+1) \\ 0; & \text{otherwise} \end{cases}$$

- *Coloring Mean*
  For $s = 5$ & $s \geq 7, q \geq 3$
  (i) If $s \equiv 0, 2 \mod 4$

$$\mu_{\chi_e}(P_s \bullet K_q) = \begin{cases} \dfrac{2q+3}{2}; & \textit{if} \quad 1 \leq i \leq 2(q+1) \end{cases}$$

(ii) If $s \equiv 1 \mod 4$

$$\mu_{\chi_e}(P_s \bullet K_q) = \begin{cases} \dfrac{q^2+7q+2}{2(q+1)}; & \textit{if} \quad i = 1; 4 \leq i \leq q+3 \\ \dfrac{3q^2+3q+4}{2(q+1)}; & \textit{if} \quad i = 2, 3; q+4 \leq i \leq 2(q+1) \end{cases}$$

(iii) If $s \equiv 3 \mod 4$

$$\mu_{\chi_e}(P_s \bullet K_q) = \begin{cases} \dfrac{q^2+7q-2}{2(q+1)}; & \textit{if} \quad i = 1, 2; 5 \leq i \leq q+3 \\ \dfrac{3q^2+3q+8}{2(q+1)}; & \textit{if} \quad i = 3, 4; q+4 \leq i \leq 2(q+1) \end{cases}$$

- *Coloring Variance*
  For $s = 5$ & $s \geq 7, q \geq 3$
  (i) If $s \equiv 0, 2 \mod 4$

$$\sigma^2_{\chi_e}(P_s \bullet K_q) = \begin{cases} \dfrac{(2q+3)(2q+1)}{12}; & \textit{if} \quad 1 \leq i \leq 2(q+1) \end{cases}$$

(ii) If $s \equiv 1 \mod 4$

$$\sigma^2_{\chi_e}(P_s \bullet K_q) = \begin{cases} \dfrac{q(q^3+4q^2+29q+74)}{12(q+1)^2}; & \textit{if} \quad i = 1; 4 \leq i \leq q+3 \\ \dfrac{q(q^3+52q^2-19q-22)}{12(q+1)^2}; & \textit{if} \quad i = 2, 3; q+4 \leq i \leq 2(q+1) \end{cases}$$

(iii) If $s \equiv 3 \mod 4$

$$\sigma^2_{\chi_e}(P_s \bullet K_q) = \begin{cases} \dfrac{q^4+4q^3+53q^2+98q-144}{12(q+1)^2}; & \textit{if} \quad i = 1, 2; 5 \leq i \leq q+3 \\ \dfrac{q(q^3+52q^2-91q+50)}{12(q+1)^2}; & \textit{if} \quad i = 3, 4; q+4 \leq i \leq 2(q+1) \end{cases}$$

Theorem 3.4. *The equitable coloring of extended corona of complete graph(K_q) with cycle graph(C_r) graph for $q, r \geq 3$ is given by*

$$\chi_{=}(K_q \bullet C_r) = \begin{cases} 3q; & \text{if } r \text{ is odd} \\ 2q; & \text{if } r \text{ is even} \end{cases}$$

Proof. Let $V(K_q) = \{t_w : 1 \leq w \leq q\}$ and $V(C_r)$=$\{g_u : 1 \leq u \leq r\}$. Let $V(K_q \bullet C_r) = \{t_w : 1 \leq w \leq q\} \bigcup \{g_{wu} : 1 \leq w \leq q, 1 \leq u \leq r\}$ be the node set of $K_q \bullet C_r$.

From the definition of Extended Corona, the graph $K_q \bullet C_r$ is obtained by taking corona of K_q with C_r and joining each vertex of i^{th} copy of C_r to every vertex of j^{th} copy of C_r provided that t_i and t_j are adjacent in K_q.

(i) **Case 1:** If r is $even$

For $1 \leq w \leq q-1$, we grant the color $2w$ to the vertices $t_{(w+1)}$ and the color $2q$ to t_1 in K_q. Now for q-copies of C_r and for $1 \leq u \leq \lceil \frac{r}{2} \rceil$, grant the color as 1 to $g_{1(2u-1)}$, 2 to $g_{1(2u)}$, 3 to $g_{2(2u-1)}$, 4 to $g_{2(2u)}$,..., $2q-1$ to $g_{q(2u-1)}$, $2q$ to $g_{q(2u)}$.

(ii) **Case 2:** If r is odd

For $1 \leq w \leq q-1$, we grant the color $3w$ to the vertices $t_{(w+1)}$ and the color $3q$ to t_1 in K_q.

Subcase 1: When $r \equiv 0 \mod 3$

For $1 \leq u \leq r$, we grant the colors 1 to g_{1u} ($u \equiv 1 \mod 3$), 2 to g_{1u} ($u \equiv 2 \mod 3$), 3 to g_{1u} ($u \equiv 0 \mod 3$), 4 to g_{2u} ($u \equiv 1 \mod 3$), 5 to g_{2u} ($u \equiv 2 \mod 3$), 6 to g_{2u} ($u \equiv 0 \mod 3$), ...,$3q-2$ to g_{qu} ($u \equiv 1 \mod 3$), $3q-1$ to g_{qu} ($u \equiv 2 \mod 3$) and $3q$ to g_{qu} ($u \equiv 0 \mod 3$) respectively.

Subcase 2: When $r \equiv 1 \mod 3$

For $1 \leq u \leq r-1$, we grant the colors 1 to g_{1u} ($u \equiv 1 \mod 3$), 2 to g_{1u} ($u \equiv 2 \mod 3$), 3 to g_{1u} ($u \equiv 0 \mod 3$), 4 to g_{2u} ($u \equiv 1 \mod 3$), 5 to g_{2u} ($u \equiv 2 \mod 3$), 6 to g_{2u} ($u \equiv 0 \mod 3$), ...,$3q-2$ to g_{qu} ($u \equiv 1 \mod 3$), $3q-1$ to g_{qu} ($u \equiv 2 \mod 3$) and $3q$ to g_{qu} ($u \equiv 0 \mod 3$) respectively. When $u = r$, we have the color $3w-1$ to g_{wu} where $1 \leq w \leq q$.

Subcase 3: When $r \equiv 2 \mod 3$

For $1 \leq u \leq r$, we grant the colors 1 to g_{1u} ($u \equiv 1 \mod 3$), 2 to g_{1u} ($u \equiv 2 \mod 3$), 3 to g_{1u} ($u \equiv 0 \mod 3$), 4 to g_{2u} ($u \equiv 1 \mod 3$), 5 to g_{2u} ($u \equiv 2 \mod 3$), 6 to g_{2u} ($u \equiv 0 \mod 3$), ...,$3q-2$ to g_{qu} ($u \equiv 1 \mod 3$), $3q-1$ to g_{qu} ($u \equiv 2 \mod 3$) and $3q$ to g_{qu} ($u \equiv 0 \mod 3$) respectively.

When r is $even$, every color $1, 3, 5, ..., 2q-1$ crop up $\left(\frac{r}{2}\right)$ times and the color $2, 4, ..., 2q$ will crop up $\left(\frac{r}{2}+1\right)$.

When r is odd

(i) $r \equiv 0 \mod 3$

The color $1, 2, 4, 5, 7, 8, ..., 3q-2, 3q-1$ will crop up $\left(\frac{r}{3}\right)$ times each and the color $3, 6, 9, ..., 3q$ will crop up $\left(\frac{r}{3}+1\right)$ times each.

(ii) $r \equiv 1 \mod 3$

The color $1, 4, 7, 10, ..., 3q-2$ will crop up $\lfloor \frac{r}{3} \rfloor$ times each and the color $2, 3, 5, 6, 9, ..., 3q-1, 3q$ will crop up $\lceil \frac{r}{3} \rceil$ times each.

(iii) $r \equiv 2 \mod 3$

The color $1, 2, 3, ..., 3q$ will crop up $\left(\frac{r+1}{3}\right)$ times each.

The absolute difference between the color classes is at most 1. The resultant graph $K_q \bullet C_r$ is $equitably\ colored$. The upper bound,

$$\chi_{=}(K_q \bullet C_r) \leq \begin{cases} 3q; & \text{if } r \text{ is odd} \\ 2q; & \text{if } r \text{ is even} \end{cases}$$

As above, we know that $\chi_{=}(K) \geq \chi(K)$ and $\chi(K_q \bullet C_r) = 2q$. And we have $\chi_{=}(K_q \bullet C_r \geq \chi(K_q \bullet C_r) = 2q$. The lower bound becomes,

$$\chi_{=}(K_q \bullet C_r) \geq 2q, \quad \text{for } q, r \geq 3.$$

Therefore, the extended corona product of K_q with C_r graph is

$$2q \leq \chi_{=}(K_q \bullet C_r) \leq \begin{cases} 3q; & \textit{if} \quad r \text{ is } odd \\ 2q; & \textit{if} \quad r \text{ is } even \end{cases}$$

□

Parameters of $\chi_{=}(K_q \bullet C_r)$

- *Probability Mass Function*

(i) If r is $even$

$$f(i) = \begin{cases} \dfrac{r}{2q(r+1)}; & \textit{if} \quad i \text{ is } odd \\ \dfrac{r+2}{2q(r+1)}; & \textit{if} \quad i \text{ is } when \\ 0; & \text{otherwise} \end{cases}$$

(ii) If r is odd

(a) $r \equiv 0 \mod 3$

$$f(i) = \begin{cases} \dfrac{r+3}{3q(r+1)}; & \textit{if} \quad i = 3, 6, 9, ..., 3q \\ \dfrac{r}{3q(r+1)}; & \textit{if} \quad i = 1, 2, 4, 5, ..., 3q-2, 3q-1 \\ 0; & \text{otherwise} \end{cases}$$

(b) $r \equiv 1 \mod 3$

$$f(i) = \begin{cases} \dfrac{r+2}{3q(r+1)}; & \textit{if} \quad i = 2, 3, 5, 6, ..., 3q-1, 3q \\ \dfrac{r-1}{3q(r+1)}; & \textit{if} \quad i = 1, 4, 7, 10, ..., 3q-2 \\ 0; & \text{otherwise} \end{cases}$$

(c) $r \equiv 2 \mod 3$

$$f(i) = \begin{cases} \dfrac{1}{3q}; & \textit{if} \quad 1 \leq i \leq 3q \\ 0; & \text{otherwise} \end{cases}$$

- *Coloring Mean*

(i) If r is $even$

$$\mu_{\chi_e}(K_q \bullet C_r) = \begin{cases} q; & \textit{if} \quad i \text{ is } odd \\ q+1; & \textit{if} \quad i \text{ is } even \end{cases}$$

(ii) If s is odd

(a) $r \equiv 0 \mod 3$

$$\mu_{\chi_e}(K_q \bullet C_r) = \begin{cases} \dfrac{3q}{2}; & \textit{if} \quad i = 1, 2, 4, 5, ..., 3q-2, 3q-1 \\ \dfrac{3(q+1)}{2}; & \textit{if} \quad i = 3, 6, 9, ..., 3q \end{cases}$$

(b) $r \equiv 1 \mod 3$

$$\mu_{\chi_e}(K_q \bullet C_r) = \begin{cases} \dfrac{3q-1}{2}; & \textit{if} \quad i = 1,4,7,9,...,3q-2 \\ \\ \dfrac{3(q+2)}{2}; & \textit{if} \quad i = 2,3,5,6,...,3q-1,3q \end{cases}$$

(c) $r \equiv 2 \mod 3$

$$\mu_{\chi_e}(K_q \bullet C_r) = \begin{cases} \dfrac{3q+1}{2}; & \textit{if} \quad i = 1,2,3,...,3q \end{cases}$$

- *Coloring Variance*

(i) If r is $even$

$$\sigma^2_{\chi_e}(P_s \bullet K_q) = \begin{cases} \dfrac{q^2-1}{3}; & \textit{if} \quad i \text{ is } odd \\ \\ \dfrac{q^2-1}{3}; & \textit{if} \quad i \text{ is } even \end{cases}$$

(i) If r is $even$

(a) If $r \equiv 0 \mod 3$

$$\sigma^2_{\chi_e}(K_q \bullet C_r) = \begin{cases} \dfrac{3q^2-2}{4}; & \textit{if} \quad i = 1,2,4,5,...,3q-2,3q-1 \\ \\ \dfrac{3(q^2-1)}{4}; & \textit{if} \quad i = 3,6,9,...,3q \end{cases}$$

(b) If $r \equiv 1 \mod 3$

$$\sigma^2_{\chi_e}(K_q \bullet C_r) = \begin{cases} \dfrac{3(q^2-1)}{4}; & \textit{if} \quad i = 1,4,7,10,...,3q-2 \\ \\ \dfrac{3q^2-2}{4}; & \textit{if} \quad i = 2,3,5,6,...,3q-1,3q \end{cases}$$

(c) If $r \equiv 2 \mod 3$

$$\sigma^2_{\chi_e}(K_q \bullet C_r) = \begin{cases} \dfrac{9q^2-1}{12}; & \textit{if} \quad 1 \leq i \leq 3q \end{cases}$$

Theorem 3.5. *The equitable coloring of extended corona of cycle graph(C_r) with complete graph(K_q) is given by*
When r is even

$$\chi_=(C_r \bullet K_q) = 2q, \ \ if \ \ q, r \geq 3$$

When r is odd

$$\chi_=(C_r \bullet K_q) = \begin{cases} 3q; & \textit{if} \quad r \equiv 0,1 \mod 3 \\ \begin{cases} 3q; & \textit{if} \quad 3 \leq r \leq r - \lfloor \frac{r}{3} \rfloor \\ 3q+1; & \textit{if} \quad otherwise \end{cases} & \textit{if} \quad r \equiv 2 \mod 3 \end{cases}$$

Proof. Let $V(C_r)$=$\{g_u : 1 \leq u \leq r\}$ and $V(K_q) = \{t_w : 1 \leq w \leq q\}$. Let $V(C_r \bullet K_q) = \{g_u : 1 \leq u \leq r\} \bigcup \{\text{t}_{uw} : 1 \leq u \leq r, 1 \leq w \leq q\}$ be the node set of $C_r \bullet K_q$.

From the definition of Extended Corona, the graph $C_r \bullet K_q$ is obtained by taking corona of C_r with K_q and joining each vertex of i^{th} copy of K_q to every vertex of j^{th} copy of K_q provided that g_i and g_j are adjacent in C_r.

(i) **Case 1**: If r is $even$

For $1 \leq u \leq \frac{r}{2}$, we grant the color w to $t_{(2u-1)w}$ and the color $q + w$ to $t_{(2u)w}$ respectively where $1 \leq w \leq q$.

Subcase 1: When $q \geq \frac{r}{2}$

For $1 \leq u \leq \frac{r}{2}$ the nodes of C_r, we grant the color u to $g_{(2u)}$ and $q + u$ to g_{2u-1}.

Subcase 2: When $q \leq \frac{r}{2}$

For $1 \leq u \leq \frac{r}{2}$, we grant the color $1, 2...q$ repeatedly to the nodes g_{2u} and $q + 1, ..., 2q$ to the nodes g_{2u-1} respectively.

(ii) **Case 2**: If r is odd

Subcase 1: When $r \equiv 0 \mod 3$

For $1 \leq u \leq r$. If $u \equiv 1 \mod 3$ we grant the color w to t_{uw}, if $u \equiv 2 \mod 3$ we grant the color $q + w$ to t_{uw}, if $u \equiv 0 \mod 3$ we grant the color $2q + w$ to t_{uw} where $1 \leq w \leq q$.

For the graph C_r and $1 \leq u \leq \frac{r}{3}$, we grant the colors $1, 2, ..., q$ repeatedly to g_u ($u \equiv 2$ mod 3), the colors $q + 1, q + 2, ..., 2q$ repeatedly to g_u ($u \equiv 0 \mod 3$), $2q + 1, 2q + 2, ..., 3q$ repeatedly to g_u ($u \equiv 1 \mod 3$).

Subcase 2: When $r \equiv 1 \mod 3$

For $1 \leq u \leq r - 1$. If $u \equiv 1 \mod 3$, we grant the colors w to t_{uw}. if $u \equiv 2 \mod 3$, we grant the colors $q + w$ to t_{uw}, if $u \equiv 0 \mod 3$, we grant the colors $2q + w$ to t_{uw} where $1 \leq w \leq q$. When $u = r$, we grant the color $q + w$ to t_{uw} where $1 \leq w \leq q$.

In the subcase 2, each of the colors $1, 2, ..., q$ have been occured $\frac{r-1}{3}$ times and the colors $q + 1, ..., 2q$ have been occured $\frac{r+2}{3}$ in the copies of K_q. While the left over colors $2q + 1, 2q + 2, ..., 3q$ has occured $\frac{r-1}{3}$ times each in copies of K_q. So, when coloring the nodes of C_r, one must use the colors $1, 2, ..., 2q$ first for g_u where $1 \leq u \leq r$ and without altering the equitable coloring condition.

Subcase 3: When $r \equiv 2 \mod 3$
(a) If $3 \leq q \leq r - \lfloor \frac{r}{3} \rfloor$

For $1 \leq u \leq r$ and $u \equiv 1 \mod 3$, we grant the color w to t_{uw}, $u \equiv 2 \mod 3$ we grant the color $q + w$ to t_{uw} and $u \equiv 0 \mod 3$ we grant the color $2q + w$ to t_{uw} where $1 \leq w \leq q$.

By above, each of the colors $1, 2, ..., 2q$ have been occured $\frac{r+1}{3}$ times in the copies of K_q. While the left over colors $2q + 1, 2q + 2, ..., 3q$ has occured $\frac{r-2}{3}$ times each in copies of K_q. So, when coloring the nodes of C_r, one must use the colors $2q + 1, 2q + 2, ..., 3q$ first for g_u where $1 \leq u \leq r$ and without altering the equitable coloring condition.

(b) **Otherwise**

For $1 \leq u \leq r$ and $u \equiv 1 \mod 3$, we grant the color w to t_{uw} and $u \equiv 2 \mod 3$ we grant the color $q + w$ to $t_{uw}$and $u \equiv 0 \mod 3$ we grant the color $2q + w$ to t_{uw} where $1 \leq w \leq q$.

For $1 \leq u \leq r$. We grant the color $2q + 1, 2q + 2, ..., 3q$ to the nodes g_u where $u \equiv 1, 2$ mod 3. When $u \equiv 0 \mod 3$, we have the color $3q + 1$.

The absolute difference between the color classes is at most 1. The resultant graph $C_r \bullet K_q$ is $equitably\ colored$. The upper bound, When r is $even$

$$\chi_{=}(C_r \bullet K_q) \leq 2q, \;\; if \;\; q, r \geq 3$$

When r is odd

$$\chi_{=}(C_r \bullet K_q) \leq \begin{cases} 3q; & if \quad r \equiv 0,1 \mod 3 \\ \begin{cases} 3q; & if \quad 3 \leq r \leq r - \lfloor \frac{r}{3} \rfloor \\ 3q+1; & if \quad otherwise \end{cases} & if \quad r \equiv 2 \mod 3 \end{cases}$$

As above, we know that $\chi_{=}(K) \geq \chi(K)$ and $\chi(C_r \bullet K_q) = 2q$. And we have $\chi_{=}(C_r \bullet K_q \geq \chi(C_r \bullet K_q) = 2q$. The lower bound becomes,

$$\chi_{=}(C_r \bullet K_q) \geq 2q, \quad \text{for } s \geq 2, q \geq 3.$$

Therefore, the extended corona product of C_r with K_q graph is
When r is $even$

$$\chi_{=}(C_r \bullet K_q) = 2q, \quad if \quad q, r \geq 3$$

When r is odd

$$2q \leq \chi_{=}(C_r \bullet K_q) \leq \begin{cases} 3q; & if \quad r \equiv 0,1 \mod 3 \\ \begin{cases} 3q; & if \quad 3 \leq r \leq r - \lfloor \frac{r}{3} \rfloor \\ 3q+1; & if \quad otherwise \end{cases} & if \quad r \equiv 2 \mod 3 \end{cases}$$

□

Conclusion

Finding $p.m.f$, $mean$, $variance$ of equitable coloring of $C_r \bullet K_q$ has so many cases in it. It's a tedious process for generalizing the occurence of each color because in some case, we colored randomly for some vertices. Hence, this part alone is an open problem to the reader.

References

[1] J. A. Bondy and U. S. R. Murthy, Graphs with Applications, SIAM Review, Vol 21 (1979).

[2] Chandrashekar Adiga, Rakshith B.R and K.N. Sudha Krishna, Spectra of Extended Neighbourhood Corona and Extended Corona of two Graphs, Electronic Journal of Graph Theory and Applications 4(1)(2016) 101-110.

[3] Dieter Jungnickel, Graphs, Networks and Algorithms, Springer- Verlag Berlin Heidelberg, Edition 4 (2013) 615-676.

[4] R. Frucht , F. Harary, On the corona of two graphs, Aequationes Math. 4 (1970), 322âĂŞ325.

[5] H. Furmanczyk, Equitable coloring of Graph products, Opuscula Mathematica, Vol 26. No.1, (2006).

[6] E. Kubicka, The chromatic sum of a graph: History and recent developments, Int. J. Math. Math. Sci., 30,(2004), 1563-1573.

[7] Lih KW. (2013) Equitable Coloring of Graphs. In: Pardalos P., Du DZ., Graham R. (eds) Handbook of Combinatorial Optimization. Springer, New York, NY.

[8] W. Meyer, Equitable coloring, Amer. Math. Monthly 80 (1973) 920âĂŞ922.

[9] Praveena, K., Venkatachalam, M., Equitable coloring on subdivision vertex join of cycle Cm with path Pn. Notes on Number Theory and Discrete Mathematics, 25(2) (2019), 190-198.

[10] Reinhard Diestel, Garph Theory, Springer- Verlag , Vol 173, Edition 5 (2017).

[11] N. K. Sudev, K. P. Chithra and J. Kok, Certain chromatic sums of some cycle related graph classes, Discrete Math. Algorithms Appl., 8(3)(2016), 1-24.

[12] Sudev N. K., Chithra K. P., Satheesh S. and Kok J., On certain parameters of equitable coloring of graphs, Discrete Math. Algorithm Appl. 9(4) (2017).

[13] N. K. Sudev, S. Satheesh, K. P. Chithra and J. Kok, On Certain Coloring Parameters of Graphs, International J.Math. Combin. Vol.3(2018), 87-98.

[14] A.C. Tucker, Perfect graphs and an application to optimizing municipal services. SIAM Rev Vol 15 (1973) 585âĂŞ590 .

Author information

M. Barani, Department of Science and Humanities, Dhanalakshmi Srinivasan College of Engineering, Coimbatore 641105, India.
E-mail: baranibe2013@gmail.com

M.Venkatachalam, PG and Research Department of Mathematics, Kongunadu Arts and Science College, Coimbatore 641029., India.
E-mail: venkatmaths@gmail.com

K.Rajalakshmi, Department of Science and Humanities, Sri Krishna College of Engineering and Technology, Coimbatore 641 008., India.
E-mail: rajalakshmikandhasamy@gmail.com

Received : December 23, 2020
Accepted : April 1, 2021