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On the Resolving Strong Domination Number of Corona and Cartesian Product of Graphs

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Abstract. The two popular research of interests in graph theory are dominating set theory and metric dimension theory. The two no tions can clearly model the real life problems and give a breakthrough to analysing a graph representation in term of distance and domination. In this paper, we try to combine the two concepts, it rises a new notion, namely a resolving strong domination set. A set $R_D \subset V(H)$ is said to be a resolving strong domination of H if R_D satisfies two conditions, namely resolving set and strong dominating set in H. The minimum cardinality of R_D such that R_D satisfies resolving set and strong dominating set is called the resolving strong domination number of H, denoted by $\gamma_{rst}(H)$. In this paper, we have obtained the exact value of the resolving strong domination number of corona and cartesian product of graphs, i.e. the corona and cartesian product of path and cycles.

1 Introduction

A graph (also known as an undirected graph or a simple graph to distinguish it from a multigraph) is a pair of $H = (V, E)$, where V is a set of vertices (singular: vertex) and E is a set of paired vertices with elements called edges (sometimes links or lines). In this study, we only use a finite, simple, un-directed and connected graph. In graph H, the vertex set V represents some elements, they could be computers, cities, bus, train, plane, etc and the edge set E represents some connection or relation between those elements. For detail definition of graph and its elements, it can be referred to Chartrand *et. al* [1].

The two popular research of interests in graph are dominating set theory and metric dimension theory. The two notions can clearly model the real life problems and give a breakthrough to analysing a graph representation in term of distance and domination. We refer to Slater [2] for the concept of resolving set of graph. Let H be a connected graph of order p and let $W = \{v_1, v_2, ..., v_k\}$ be an ordered set of vertices of G. For a vertex u of G, the k-vector $r(u|W) = (d(u, v_1), d(u, v_2), ..., d(u, v_k))$, where $d(u, v)$ represents the distance between the vertices u and v, is called the representation of vertices with respect to W. The set W is a resolving set for H if $r(u|W) = r(v|W)$ implies that $u = v$ for every pair u, v of vertices of H. A resolving set of minimum cardinality is called a minimum resolving set. The minimum cardinality of resolving set of H is its *dimension* of H, denoted by $dim(H)$. The concepts of resolving set and minimum resolving set have previously appeared in the literature [2, 3].

A subset D of the vertex set V of a graph H is said to be a dominating set of H if every vertex in $D - V$ is adjacent to a vertex in D . The minimum cardinality of a dominating set is called the domination number of H and is denoted by $\gamma(H)$[4]. In this paper we study a variant of this classical notion, namely the strong domination. A set $D \subseteq V$ is called a strong dominating set if for every vertex $v \in V - D$, there exists a vertex $u \in D$ such that $uv \in E(H)$ and $deg(u) \geq deg(v)$. The minimum cardinality of a strong dominating set is called the strong domination number of H and is denoted by $\gamma_{st}(H)$ [5, 6].

We initiate to study a new notion, namely the combination resolving set and strong dominating set. A set $R_D \subset V(H)$ is said to be a resolving strong domination of H if R_D satisfies two conditions, namely resolving set and strong dominating set in H. The minimum cardinality of R_D such that R_D satisfies resolving set and strong dominating set is called the resolving strong domination number of H, denoted by $\gamma_{rst}(H)$. Dafik et al [10] have obtained the bound of the resolving strong domination number of any graph. Furthermore, Wardani *et al* [7, 8, 9] have determined some results on the domination number of some graphs, while Dafik *et al* have studied the resolving domination number of some graphs in [10, 11].

Lemma 1.1. *[10] The strong domination number of any graph H satistisfies*

$$max\{\gamma_{st}(H), dim(H)\} \leq \gamma_{rst}(H) \leq min\{\gamma_{st}(H) + dim(H), |V(H)| - 1\}.$$

Now, we recall the definition of corona and cartesian product of graphs. Let H_1 and H_2 be two graphs of order n and m, respectively. The corona product of graph H_1 and H_2, denoted by $H_1 \odot H_2$ is defined as the graph obtained from H_1 and H_2 by taking one copy of H_1 and n copies of H_2 and joining by an edge each vertex from the ith-copy of H_2 with the ith-vertex of H_1. While the cartesian product of H_1 and H_2 denoted by $H_1 \times H_2$ is the graph of order $n \times m$ with vertex set $V(H_1) \times V(H_2) = \{(x, y)|x \in V(H_1)\ and\ y \in V(H_2)\}$ such that two vertices $(x, y), (x', y')$ are adjacent if only if either $x = x'$ in H_1 and yy' in $E(H_2)$ or xx' in $E(H_1)$ and $y = y'$ in H_2. Iswadi êt. al stated the exact value of dimension number of corona product of any two graphs as follows.

Theorem 1.2. *[12] Let G, H be connected graph, with H with order of at least 2. The dimension number of $dim(G \odot H)$ is*

$$dim(G \odot H) = \begin{cases} |G|dim(H), & \textit{if } H \textit{ contains a dominant vertex;} \\ |G|dim(K_1 + H), & \textit{otherwise} \end{cases}$$

From now on, we start to give our result on resolving strong domination number in the following sections.

2 The Resolving Strong Domination Number of Corona Product Graphs

We have obtained the resolving strong domination number of the corona product of graphs, i.e. path and cycle, denoted by $\gamma_{rst}(P_n \odot P_m)$.

Theorem 2.1. *For every positive integer $n, m \geq 3$,*

$$\gamma_{rst}(P_n \odot P_m) = n\left\lceil \frac{m}{2} \right\rceil.$$

Proof. Graph $P_n \odot P_m$ is a connected graph with vertex set $V(P_n \odot P_m) = \{x_i; 1 \leq i \leq n\} \cup \{y_j^i; 1 \leq j \leq m, 1 \leq i \leq n\}$ and edge set $E(P_n \odot P_m) = \{x_i x_{i+1}; 1 \leq i \leq n-1\} \cup \{y_j^i y_{j+1}^i; 1 \leq j \leq m-1, 1 \leq i \leq n\} \cup \{x_i y_j^i; 1 \leq j \leq m, 1 \leq i \leq n\}$. The cardinality of vertex set $V(P_n \odot P_m)$ is $n + m$ and the cardinality of edge set $E(P_n \odot P_m)$ is $2nm - 1$.

We divide two cases to show the proof. First, determining the lower bound and upper bound of $\gamma_{rst}(P_n \odot P_m)$. We use Lemma 1.1 to show the lower bound of $\gamma_{rst}(P_n \odot P_m)$, thus we have $\gamma_{rst}(P_n \odot P_m) \geq max\{\gamma_{st}(P_n \odot P_m), dim(P_n \odot P_m)\}$. Based on Theorem 1.2 we know that $dim(P_n \odot P_m) = n\ dim(K_1 + P_m) = n\lceil \frac{m}{2} \rceil$. It is easy to see that $\gamma_{st}(P_n \odot P_m) = n$. Hence, it implies

$$\begin{aligned} \gamma_{rst}(P_n \odot P_m) &\geq max\{\gamma_{st}(P_n \odot P_m), dim(P_n \odot P_m)\} \\ &= max\{n, n\lceil \tfrac{m}{2} \rceil\} = n\left\lceil \tfrac{m}{2} \right\rceil \end{aligned}$$

Furthermore, we determine the upper bound of $\gamma_{rst}(P_n \odot P_m)$ by defining the resolving strong dominating set $R_D(P_n \odot P_m)$. By considering the above vertex and edge sets, we define the following resolving strong dominating set $R_D = \{x_i, y_j^i; 1 \leq i \leq n, 2 \leq j \leq m-1\ and\ j \equiv 0 (mod 2)\}$.

Secondly, we need to show the distinction of $r(u|R_D)$ showing the distance between the vertices $u \in V(P_n \odot P_m)$ and $v \in R_D$. They are all different and it can be shown in Table 1. It concludes the proof. □

Table 1. The distinction of $r(u|R_D)$ where $u \in V(P_n \odot P_m)$

	Resolving Strong Dominating Set (R_D)																						
$V(P_n \odot P_m)$	x_1	x_2	x_3	$\cdots$	x_n	y_2^1	y_4^1	y_6^1	$\cdots$	y_2^2	y_4^2	y_6^2	$\cdots$	y_2^3	y_4^3	y_6^3	$\cdots$	$\cdots$	y_2^n	y_4^n	y_6^n	$\cdots$
x_1	0	1	2	$\cdots$	n-1	1	1	1	$\cdots$	2	2	2	$\cdots$	3	3	3	$\cdots$	$\cdots$	n	n	n	$\cdots$
x_2	1	0	1	$\cdots$	n-2	2	2	2	$\cdots$	1	1	1	$\cdots$	2	2	2	$\cdots$	$\cdots$	n-1	n-1	n-1	$\cdots$
x_3	2	1	0	$\cdots$	n-3	3	3	3	$\cdots$	2	2	2	$\cdots$	1	1	1	$\cdots$	$\cdots$	n-2	n-2	n-2	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_n	n-1	n-2	n-3	$\cdots$	0	n	n	n	$\cdots$	n-1	n-1	n-1	$\cdots$	n-2	n-2	n-2	$\cdots$	$\cdots$	1	1	1	$\cdots$
y_1^1	1	2	3	$\cdots$	n	1	2	2	$\cdots$	3	3	3	$\cdots$	4	4	4	$\cdots$	$\cdots$	n+1	n+1	n+1	$\cdots$
y_2^1	1	2	3	$\cdots$	n	0	2	2	$\cdots$	3	3	3	$\cdots$	4	4	4	$\cdots$	$\cdots$	n+1	n+1	n+1	$\cdots$
y_3^1	1	2	3	$\cdots$	n	1	1	2	$\cdots$	3	3	3	$\cdots$	4	4	4	$\cdots$	$\cdots$	n+1	n+1	n+1	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
y_m^1	1	2	3	$\cdots$	n	2	2	2	$\cdots$	3	3	3	$\cdots$	4	4	4	$\cdots$	$\cdots$	n+1	n+1	n+1	$\cdots$
y_1^2	2	1	2	$\cdots$	n-1	n-2	3	3	$\cdots$	1	2	2	$\cdots$	3	3	3	$\cdots$	$\cdots$	n	n	n	$\cdots$
y_2^2	2	1	2	$\cdots$	n-1	n-2	3	3	$\cdots$	0	2	2	$\cdots$	3	3	3	$\cdots$	$\cdots$	n	n	n	$\cdots$
y_3^2	2	1	2	$\cdots$	n-1	n-2	3	3	$\cdots$	1	1	2	$\cdots$	3	3	3	$\cdots$	$\cdots$	n	n	n	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
y_m^2	2	1	2	$\cdots$	n-1	n-2	3	3	$\cdots$	2	2	2	$\cdots$	3	3	3	$\cdots$	$\cdots$	n	n	n	$\cdots$
y_1^3	3	2	1	$\cdots$	n-2	n-2	4	4	$\cdots$	3	3	3	$\cdots$	1	2	2	$\cdots$	$\cdots$	n-1	n-1	n-1	$\cdots$
y_2^3	3	2	1	$\cdots$	n-2	4	4	4	$\cdots$	3	3	3	$\cdots$	0	2	2	$\cdots$	$\cdots$	n-1	n-1	n-1	$\cdots$
y_3^3	3	2	1	$\cdots$	n-2	4	4	4	$\cdots$	3	3	3	$\cdots$	1	1	2	$\cdots$	$\cdots$	n-1	n-1	n-1	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
y_m^3	3	2	1	$\cdots$	n-2	4	4	4	$\cdots$	3	3	3	$\cdots$	2	2	2	$\cdots$	$\cdots$	n-1	n-1	n-1	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
y_1^n	n	n-1	n-2	$\cdots$	1	n+1	n+1	n+1	$\cdots$	n	n	n	$\cdots$	n-1	n-1	n-1	$\cdots$	$\cdots$	1	2	2	$\cdots$
y_2^n	n	n-1	n-2	$\cdots$	1	n+1	n+1	n+1	$\cdots$	n	n	n	$\cdots$	n-1	n-1	n-1	$\cdots$	$\cdots$	0	2	2	$\cdots$
y_3^n	n	n-1	n-2	$\cdots$	1	n+1	n+1	n+1	$\cdots$	n	n	n	$\cdots$	n-1	n-1	n-1	$\cdots$	$\cdots$	1	1	2	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
y_m^n	n	n-1	n-2	$\cdots$	1	n+1	n+1	n+1	$\cdots$	n	n	n	$\cdots$	n-1	n-1	n-1	$\cdots$	$\cdots$	2	2	2	$\cdots$

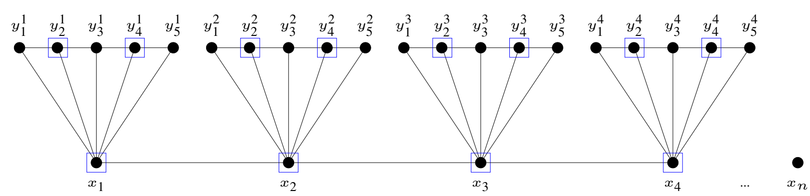

Figure 1. The illustration of the resolving strong dominating set of $P_4 \odot P_5$

For more detail explanation, we give an illustration of the resolving strong domination of $P_4 \odot P_5$ in Figure 1. Based on the Figure 1, the resolving strong dominating set $R_D(P_4 \odot P_5) = \{x_1, x_2, x_3, x_4, y_2^1, y_4^1, y_2^2, y_4^2, y_2^3, y_4^3, y_2^4, y_4^4\}$. Thus $\gamma_{rst}(P_4 \odot C_5) = 4\left\lceil \frac{5}{2} \right\rceil = 12$. We present the $r(u|R_D)$ showing the distance between the vertices $u \in V(P_4 \odot P_5)$ and $v \in R_D$ in the following.

$$
\begin{aligned}
x_1 &= (0,1,2,3,1,1,2,2,3,3,4,4), \; x_2 = (1,0,1,2,2,2,1,1,2,2,3,3),\\
x_3 &= (2,1,0,1,3,3,2,2,1,1,2,2), \; x_4 = (2,1,0,1,3,3,2,2,1,1,2,2),\\
y_1^1 &= (1,2,3,4,1,2,3,3,4,4,5,5), \; y_2^1 = (1,2,3,4,0,2,3,3,4,4,5,5),\\
y_3^1 &= (1,2,3,4,1,1,3,3,4,4,5,5), \; y_4^1 = (1,2,3,4,2,0,3,3,4,4,5,5),\\
y_5^1 &= (1,2,3,4,2,1,3,3,4,4,5,5), \; y_1^2 = (2,1,2,3,3,3,1,2,3,3,4,4),\\
y_2^2 &= (2,1,2,3,3,3,0,2,3,3,4,4), \; y_3^2 = (2,1,2,3,3,3,1,1,3,3,4,4),\\
y_4^2 &= (2,1,2,3,3,3,2,0,3,3,4,4), \; y_5^2 = (2,1,2,3,3,3,2,1,3,3,4,4),\\
y_1^3 &= (3,2,1,2,4,4,3,3,1,2,3,3), \; y_2^3 = (3,2,1,2,4,4,3,3,0,2,3,3),\\
y_3^3 &= (3,2,1,2,4,4,3,3,1,1,3,3), \; y_4^3 = (3,2,1,2,4,4,3,3,2,0,3,3),\\
y_5^3 &= (3,2,1,2,4,4,3,3,2,1,3,3), \; y_1^4 = (4,3,2,1,5,5,4,4,3,3,1,2),\\
y_2^4 &= (4,3,2,1,5,5,4,4,3,3,0,2), \; y_3^4 = (4,3,2,1,5,5,4,4,3,3,1,1),\\
y_4^4 &= (4,3,2,1,5,5,4,4,3,3,2,0), \; y_5^4 = (4,3,2,1,5,5,4,4,3,3,2,1).
\end{aligned}
$$

Theorem 2.2. *For every positive integer $n, m \geq 3$,*

$$\gamma_{rst}(P_n \odot C_m) = n\left\lceil \frac{m}{2} \right\rceil.$$

Proof. Graph $P_n \odot C_m$ is a connected graph with vertex set $V(P_n \odot C_m) = \{x_i; 1 \leq i \leq n\} \cup \{y_j^i; 1 \leq j \leq m, 1 \leq i \leq n\}$ and edge set $E(P_n \odot C_m) = \{x_i x_{i+1}; 1 \leq i \leq n-1\} \cup \{y_j^i y_{j+1}^i; 1 \leq$

$j \leq m-1, 1 \leq i \leq n\} \cup \{y_i^i y_m^i; 1 \leq i \leq n\} \cup \{x_i y_j^i; 1 \leq j \leq m, 1 \leq i \leq n\}$. The cardinality of vertex set $V(P_n \odot C_m)$ is $n+m$ and the cardinality of edge set $E(P_n \odot C_m)$ is $2nm+n-1$.

To prove this theorem, We divide two cases, namely determining the lower bound and upper bound of $\gamma_{rst}(P_n \odot C_m)$. We use Lemma 1.1 to show the lower bound of $\gamma_{rst}(P_n \odot C_m)$, thus we have $\gamma_{rst}(P_n \odot C_m) \geq max\{\gamma_{st}(P_n \odot C_m), dim(P_n \odot C_m)\}$. Based on Theorem 1.2 we know that $dim(P_n \odot C_m) = n\, dim(K_1 + C_m) = n\lceil \frac{m}{2} \rceil$. It is easy to see that $\gamma_{st}(P_n \odot C_m) = n$. Hence, it implies

$$\begin{aligned}\gamma_{rst}(P_n \odot C_m) &\geq max\{\gamma_{st}(P_n \odot C_m), dim(P_n \odot C_m)\}\\ &= max\{n, n\lceil \tfrac{m}{2} \rceil\} = n\left\lceil \frac{m}{2} \right\rceil\end{aligned}$$

Furthermore, we determine the upper bound of $\gamma_{rst}(P_n \odot C_m)$ by defining the resolving strong dominating set $R_D(P_n \odot C_m)$. By considering the above vertex and edge sets, we define the following resolving strong dominating set $R_D = \{x_i, y_j^i; 1 \leq i \leq n, 2 \leq j \leq m-1 \ and\ j \equiv 0(mod 2)\}$.

Secondly, we need to show the distinction of $r(u|R_D)$ showing the distance between the vertices $u \in V(P_n \odot C_m)$ and $v \in R_D$. They are all different and it can be shown in Table 1. It completes the proof. □

3 The Resolving Strong Domination Number of Cartesian Product Graphs

In this section, We show two theorems on the resolving strong domination of the cartesian product of graphs, namely path and cycle. The definition of this graph are shown in the introduction.

Theorem 3.1. *For every positive integer $n, m \geq 3$ and $m \geq n$,*

$$\gamma_{rst}(P_n \times P_m) = \begin{cases} n+1, & if\ m = 3 \\ \lceil \frac{m}{3} \rceil n, & otherwise \end{cases}$$

Proof. Path graph, namely P_n and P_m have vertex set $V(P_n) = \{u_1, u_2, u_3, \ldots, u_n$ and $V(P_m) = \{v_1, v_2, v_3, \ldots, v_m\}$, respectively. Graph $P_n \times P_m$ is a connected graph with vertex set $\{(u,v)|u \in V(P_n)$ and $v \in V(P_m)$, such that two vertices $(u_1, v_1), (u_2, v_2)$ are adjacent if only if either $u_1 = u_2$ in P_n and $v_1 v_2$ in $E(P_m)$ or $u_1 u_2$ in $E(P_n)$ and $v_1 = v_2$ in P_m. We divide two cases to show the proof.

Case 1. For $m = 3$.

First, determining the lower bound and upper bound of $\gamma_{rst}(P_n \times P_3)$. We use Lemma 1.1 to show the lower bound of $\gamma_{rst}(P_n \times P_3)$, thus we have $\gamma_{rst}(P_n \times P_3) \geq max\{\gamma_{st}(P_n \times P_3), dim(P_n \times P_3)\}$. We know that $\gamma_{st}(P_n \times P_3) = n$ and $dim(P_n \times P_3) = 2$. Hence, it implies

$$\begin{aligned}\gamma_{rst}(P_n \times P_3) &\geq max\{\gamma_{st}(P_n \times P_3), dim(P_n \times P_3)\}\\ &= max\{n, 2\} = n\end{aligned}$$

Furthermore, we determine the upper bound of $\gamma_{rst}(P_n \times P_3)$ by defining the resolving strong dominating set $R_D(P_n \times P_3)$. Suppose $R_D(P_n \times P_3) = \{(u_i, v_2) : 1 \leq i \leq n\}$ and we illustrate this resolving strong dominating set in Figure 2. Based on the illustration in Figure 2, the vertices (u_i, v_1) and (u_i, v_3) will receive the same representation for $1 \leq i \leq n$. Hence, we add 1 vertex to the resolving strong dominating set $R_D(P_n \times P_3)$, such that the representations of vertices in $V(P_n \times P_3)$ are all distinct. The resolving strong dominating set of $(P_n \times P_3)$ is $R_D(P_n \times P_3) = \{(u_2, v_1), (u_i, v_2) : 1 \leq i \leq n\}$, see Figure 3 to this illustration. Based on the Figure 3, we give the all representation of each vertex in $V(P_n \times P_3)$ in the following.

$$\begin{aligned}(u_1, v_1) &= (1, 1, 2, 3, ..., n-2, n-1, n)\\ (u_2, v_1) &= (0, 2, 1, 2, ..., n-3, n-2, n-1)\\ (u_3, v_1) &= (1, 3, 2, 1, ..., n-4, n-3, n-2)\\ &\vdots\end{aligned}$$

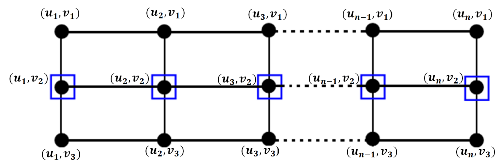

Figure 2. The illustration of the resolving strong dominating set of $P_n \odot P_3$.

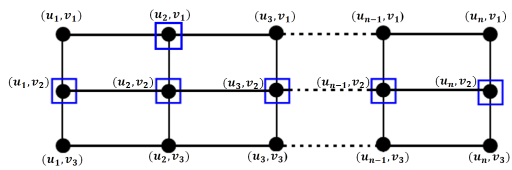

Figure 3. The illustration of the resolving strong dominating set of $P_n \odot P_3$.

$$
\begin{gathered}
(u_{n-1}, v_1) = (n-3, n-1, n-2, n-3, ..., 2, 1, 2) \\
(u_n, v_1) = (n-2, n, n-1, n-2, ..., 3, 2, 1) \\
(u_1, v_2) = (2, 0, 1, 2, ..., n-3, n-2, n-1) \\
(u_2, v_2) = (1, 1, 0, 1, ..., n-4, n-3, n-2) \\
(u_3, v_2) = (2, 2, 1, 0, ..., n-5, n-4, n-3) \\
\vdots \\
(u_{n-1}, v_2) = (n-2, n-2, n-3, n-4, ..., 3, 2, 1) \\
(u_n, v_2) = (n-1, n-1, n-2, n-3, ..., 2, 1, 0)
\end{gathered}
$$

$$
\begin{gathered}
(u_1, v_3) = (3, 1, 2, 3, ..., n-2, n-1, n) \\
(u_2, v_3) = (2, 2, 1, 2, ..., n-3, n-2, n-1) \\
(u_3, v_3) = (3, 3, 2, 1, ..., n-4, n-3, n-2) \\
\vdots \\
(u_{n-1}, v_3) = (n-1, n-1, n-2, n-3, ..., 2, 1, 2) \\
(u_n, v_3) = (n, n, n-1, n-2, ..., 3, 2, 1)
\end{gathered}
$$

It It completes the proof that $\gamma_{rst}(P_n \times P_3) = n + 1$ for $m = 3$. □

Case 2. For m otherwise.
First, determining the lower bound and upper bound of $\gamma_{rst}(P_n \times P_m)$. We use Lemma 1.1 to show the lower bound of $\gamma_{rst}(P_n \times P_m)$, thus we have $\gamma_{rst}(P_n \times P_m) \geq max\{\gamma_{st}(P_n \times P_m), dim(P_n \times P_m)\}$. We know that $\gamma_{st}(P_n \times P_m) = \left\lceil \frac{m}{3} \right\rceil n$ and $dim(P_n \times P_m) = 2$. Hence, it implies

$$
\begin{aligned}
\gamma_{rst}(P_n \times P_m) &\geq max\{\gamma_{st}(P_n \times P_m), dim(P_n \times P_m)\} \\
&= max\{\left\lceil \frac{m}{3} \right\rceil n, 2\} = \left\lceil \frac{m}{3} \right\rceil n
\end{aligned}
$$

Furthermore, we determine the upper bound of $\gamma_{rst}(P_n \times P_m)$ by defining the resolving strong dominating set $R_D(P_n \times P_m)$. We divide three cases for defining the resolving strong dominating set of $(P_n \times P_m)$, namely

(i) For $m \equiv 0(mod)$, $R_D(P_n \times P_m) = \{(u_l, v_k) : 1 \leq l \leq n, 1 \leq k \leq m \ and \ k \equiv 2(mod\ 3)\}$

(ii) For $m \equiv 2(mod)$, $R_D(P_n \times P_m) = \{(u_l, v_{m-1}), (u_l, v_k) : 1 \leq l \leq n, 1 \leq k < m \ and \ k \equiv 2(mod\ 3)\}$

(iii) For $m \equiv 1(mod)$, $R_D(P_n \times P_m) = \{(u_l, v_{m-1}), (u_l, v_k) : 1 \leq l \leq n, 1 \leq k < m - 1 \ and \ k \equiv 2(mod\ 3)\}$

Let the representations of each vertex in $V(P_n \times P_m)$ to R_D is $r[(u_i,v_j)|R_D] = (\alpha_l^k : 1 \leq l \leq n, 1 \leq k \leq m \; and \; k \equiv 2(mod\; 3))$. α_l^k is a distance of a vertex $(u_i,v_j) \in V(P_n \times P_m)$ to every vertices in $R_D(P_n \times P_m)$ by the function $f : d[(u_i,v_j), R_D(P_n \times P_m)] \longrightarrow \alpha_l^k$, where

$$\alpha_l^k = \begin{cases} l - i + |j-k|, & \text{if } i \leq l \\ i - l + |j-k|, & \text{if } i > l \end{cases}$$

It easy to see that the representations of vertices in $V(P_n \times P_3)$ are all distinct. It It completes the proof that $\gamma_{rst}(P_n \times P_3) = \left\lceil \frac{m}{3} \right\rceil n$ for m otherwise. □

Theorem 3.2. *For every positive integer $n, m \geq 3$ and $m \geq n$,*

$$\gamma_{rst}(P_n \times C_m) = \begin{cases} n+1, & \textit{if } m = 3 \\ \lceil \frac{m}{3} \rceil n, & \textit{otherwise} \end{cases}$$

Let P_n and C_m are path and cycle graph, respectively. The vertex set of path is $V(P_n) = \{u_1, u_2, u_3, \ldots, u_n$ and vertex set of cycle is $V(C_m) = \{v_1, v_2, v_3, \ldots, v_m\}$. Graph $P_n \times C_m$ is a connected graph with vertex set $\{(u,v)|u \in V(P_n)$ and $v \in V(C_m)$, such that two vertices $(u_1,v_1), (u_2,v_2)$ are adjacent if only if either $u_1 = u_2$ in P_n and v_1v_2 in $E(C_m)$ or u_1u_2 in $E(P_n)$ and $v_1 = v_2$ in C_m. We divide into two cases to prove the resolving strong domination number of $P_n \times C_m$.

Case 1. For $m = 3$.

First, determining the lowerbound and upperbound of $\gamma_{rst}(P_n \times C_3)$. We use Lemma 1.1 to show the lowerbound of $\gamma_{rst}(P_n \times C_3)$, thus we have $\gamma_{rst}(P_n \times C_3) \geq max\{\gamma_{st}(P_n \times C_3), dim(P_n \times C_3)\}$. We know that $\gamma_{st}(P_n \times C_3) = n$ and $dim(P_n \times C_3) = 2$. Hence, it implies

$$\begin{aligned} \gamma_{rst}(P_n \times C_3) &\geq max\{\gamma_{st}(P_n \times C_3), dim(P_n \times C_3)\} \\ &= max\{n, 2\} = n \end{aligned}$$

Furthermore, we determine the upperbound of $\gamma_{rst}(P_n \times C_3)$ by defining the resolving strong dominating set $R_D(P_n \times C_3)$. By considering the above vertex and edge sets, we define resolving strong dominating set $R_D(P_n \times C_3) = \{(u_i, v_2) : 1 \leq i \leq n\}$ and we illustrate this resolving strong dominating set in Figure 4. Based on the illustration in Figure 4, the vertices (u_i, v_1) and (u_i, v_3) will receive the same representation for $1 \leq i \leq n$. Hence, we add 1 vertex to the resolving strong dominating set $R_D(P_n \times C_3)$, such that the representations of vertices in $V(P_n \times C_3)$ are all distinct. The resolving strong dominating set of $(P_n \times C_3)$ is $R_D(P_n \times C_3) = \{(u_2, v_1), (u_i, v_2) : 1 \leq i \leq n\}$, see Figure 5 to this illustration. Based on the Figure 5, we give the all representation of each vertex in $V(P_n \times C_3)$ in the following.

$$\begin{aligned}
(u_1, v_1) &= (1, 1, 2, 3, ..., n-2, n-1, n) \\
(u_2, v_1) &= (0, 2, 1, 2, ..., n-3, n-2, n-1) \\
(u_3, v_1) &= (1, 3, 2, 1, ..., n-4, n-3, n-2) \\
&\vdots \\
(u_{n-1}, v_1) &= (n-3, n-1, n-2, n-3, ..., 2, 1, 2) \\
(u_n, v_1) &= (n-2, n, n-1, n-2, ..., 3, 2, 1) \\
(u_1, v_2) &= (2, 0, 1, 2, ..., n-3, n-2, n-1) \\
(u_2, v_2) &= (1, 1, 0, 1, ..., n-4, n-3, n-2) \\
(u_3, v_2) &= (2, 2, 1, 0, ..., n-5, n-4, n-3) \\
&\vdots \\
(u_{n-1}, v_2) &= (n-2, n-2, n-3, n-4, ..., 3, 2, 1) \\
(u_n, v_2) &= (n-1, n-1, n-2, n-3, ..., 2, 1, 0)
\end{aligned}$$

$$\begin{aligned}
(u_1, v_3) &= (2, 1, 2, 3, ..., n-2, n-1, n) \\
(u_2, v_3) &= (1, 2, 1, 2, ..., n-3, n-2, n-1) \\
(u_3, v_3) &= (2, 3, 2, 1, ..., n-4, n-3, n-2)
\end{aligned}$$

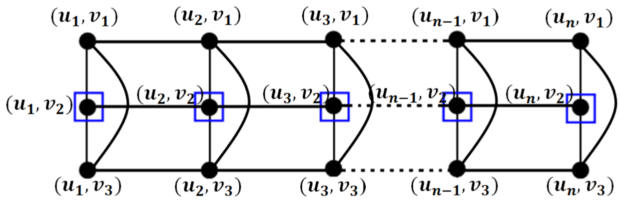

Figure 4. The illustration of the resolving strong dominating set of $P_n \odot C_3$.

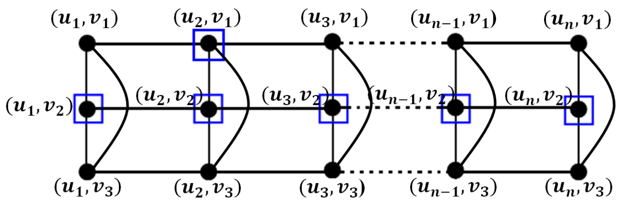

Figure 5. The illustration of the resolving strong dominating set on $P_n \odot C_3$.

$$\vdots$$
$$(u_{n-1}, v_3) = (n-2, n-1, n-2, n-3, ..., 2, 1, 2)$$
$$(u_n, v_3) = (n-1, n, n-1, n-2, ..., 3, 2, 1)$$

It completes the proof that $\gamma_{rst}(P_n \times P_3) = n + 1$ for $m = 3$.□
Case 2. For m otherwise.
First, determining the lowerbound and upperbound of $\gamma_{rst}(P_n \times C_m)$. We use Lemma 1.1 to show the lowerbound of $\gamma_{rst}(P_n \times C_m)$, thus we have $\gamma_{rst}(P_n \times C_m) \geq max\{\gamma_{st}(P_n \times C_m), dim(P_n \times C_m)\}$. We know that $\gamma_{st}(P_n \times C_m) = \left\lceil \frac{m}{3} \right\rceil n$ and $dim(P_n \times C_m) = 2$. Hence, it implies

$$\begin{aligned}\gamma_{rst}(P_n \times C_m) &\geq max\{\gamma_{st}(P_n \times C_m), dim(P_n \times C_m)\}\\ &= max\{\left\lceil \frac{m}{3} \right\rceil n, 2\} = \left\lceil \frac{m}{3} \right\rceil n\end{aligned}$$

Furthermore, we determine the upper bound of $\gamma_{rst}(P_n \times C_m)$ by defining the resolving strong dominating set $R_D(P_n \times C_m)$. We divide three cases for defining the resolving strong dominating set of $(P_n \times C_m)$, namely

(i) For $m \equiv 0(mod)$, $R_D(P_n \times C_m) = \{(u_l, v_k) : 1 \leq l \leq n, 1 \leq k \leq m \; and \; k \equiv 2(mod\ 3)\}$

(ii) For $m \equiv 2(mod)$, $R_D(P_n \times C_m) = \{(u_l, v_{m-1}), (u_l, v_k) : 1 \leq l \leq n, 1 \leq k < m \; and \; k \equiv 2(mod\ 3)\}$

(iii) For $m \equiv 1(mod)$, $R_D(P_n \times C_m) = \{(u_l, v_{m-1}), (u_l, v_k) : 1 \leq l \leq n, 1 \leq k < m - 1 \; and \; k \equiv 2(mod\ 3)\}$

Let the representations of each vertex in $V(P_n \times C_m)$ to R_D is $r[(u_i, v_j)|R_D] = (\alpha_l^k : 1 \leq l \leq n, 1 \leq k \leq m \; and \; k \equiv 2(mod\ 3))$. α_l^k is a distance of a vertex $(u_i, v_j) \in V(P_n \times C_m)$ to every vertices in $R_D(P_n \times C_m)$ by the function $f : d[(u_i, v_j), R_D(P_n \times C_m)] \longrightarrow \alpha_l^k$, where

$$\alpha_l^k = \begin{cases} l-i+|j-k|, & \text{if } i \le l, k < n \text{ and } j < (k+n) \text{ or} \\ & k > n \text{ and } j > (k+n) \text{mod } m \\ i-l+|j-k|, & \text{if } i > l, k < n \text{ and } j < (k+n) \text{ or} \\ & k > n \text{ and } j > (k+n) \text{mod } m \\ l-i+2n-|j-k|-1, & \text{if } i \le l, k < n \text{ and } j \ge (k+n) \text{ or} \\ & k > n \text{ and } j \le (k+n) \text{mod } m \\ i-l+2n-|j-k|-1, & \text{if } i > l, k < n \text{ and } j \ge (k+n) \text{ or} \\ & k > n \text{ and } j \le (k+n) \text{mod } m \end{cases}$$

It easy to see that the representations of vertices in $V(P_n \times C_m)$ are all distinct. It completes the proof that $\gamma_{rst}(P_n \times C_3) = \left\lceil \frac{m}{3} \right\rceil n$ for m otherwise. □

4 Concluding Remark

The results in this paper are finding the exact values of $\gamma_{rst}(H)$, where H are $P_n \odot P_m$, $P_n \odot C_m$, $P_n \times P_m$ and $P_n \times C_m$. However, to determine γ_{rst} of any graph H is considered to be an NP-problem. Therefore we propose the following open problems.

(i) Determine γ_{rst} of any graph H apart from above investigated graphs.

(ii) Determine the sharpest lower and upper bound of γ_{rst} for any coronation and cartesian product of graphs.

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