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SOME SEPERATION AXIOMS ON $\mathcal{N}s\widehat{g}$-CLOSED SETS

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Abstract The basic objective of this paper is to introduce and look into the properties of $\mathcal{N}s\widehat{g}$-T_0 space, $\mathcal{N}s\widehat{g}$-T_1 space, $\mathcal{N}s\widehat{g}$-T_2, $\mathcal{N}s\widehat{g}$-regular and $\mathcal{N}s\widehat{g}$-normal spaces and obtain the relation between some of the subsisting sets.

1 Introduction

Lellis Thivagar and Richard [1] established the notion of nano topology in terms of approximations and boundary region of a subset of an universe using an equivalence relation on it and also make known about nano-closed sets, nano-interior, nano-closure and weak form of nano open sets namely nano semi-open sets, nano pre-open, nano α-open sets and nano semi pre-open sets. Nasef et.al.[2] make known about some of nearly open sets in nano topological spaces. Revathy and Gnanambal Illango [4] gave the idea about the nano β-open sets. Sathishmohan et.al.[6] brings up the idea about nano neighourhoods and study the properties of nano semi pre-T_0 space, nano semi pre-T_1 space, nano semi pre T_2-space in nano topological spaces.

2 Preliminaries

Definition 2.1. [3] Let U be a non-empty finite set of objects called the universe and R be an equivalence relation on U named as indiscernibility relation. Then U is divided into disjoint equivalence classes. Elements belonging to the same equivalence class are said to be indiscernible with one another. The pair (U, R) is said to be the approximation space. Let $X \subseteq U$. Then,
(i)The lower approximation of x with respect to R is the set of all objects, which can be for certain classified as X with respect to R and is denoted by $L_R(X)$.
$L_R(X) = \bigcup_{x\in U}\{R(x) : R(x) \subseteq X\}$ where $R(x)$ denotes the equivalence class determined by $x \in U$.
(ii)The upper approximation of x with respect to R is the set of all objects which can be possibly classified as X with respect to R and is denoted by $U_R(X)$.
$U_R(X) = \bigcup_{x\in U}\{R(x) : R(x) \cap X \neq \phi\}$
(iii)The boundary region of x with respect to R is the set of all objects which can be classified neither as X nor as not-X with respect to R and it is denoted by $B_R(X)$.
$B_R(X) = U_R(X)$ - $L_R(X)$.

Definition 2.2. [3] Let U be the universe, R be an equivalence relation on U and $\tau_R(X) = \{U, \phi, L_R(X), U_R(X), B_R(X)\}$ where $X \subseteq U$. Then $\tau_R(X)$ satisfies the following axioms
(i) U and $\phi \in \tau_R(X)$.
(ii) The union of the elements of any sub-collection of $\tau_R(X)$ is in $\tau_R(X)$.
(iii) The intersection of the elements of any finite sub collection of $\tau_R(X)$ is in $\tau_R(X)$.
Then $\tau_R(X)$ is a topology on U called the nano topology on U with respect to X. We call $(U, \tau_R(X))$ as nano topological space. The elements of $\tau_R(X)$ are called as nano-open sets. The complement of the nano-open sets are called nano-closed sets.

Remark 2.3. [3] If $\tau_R(X)$ is the nano topology on U with respect to X , then the set $B = \{U, L_R(X), B_R(X)\}$ is the basis for $\tau_R(X)$.

Definition 2.4. [1] If $(U, \tau_R(X))$ is a nano topological space with respect to X where $X \subseteq U$ and if $A \subseteq U$, then
(i)The nano interior of A is defined as the union of all nano-open subsets of A is contained in A and is denoted by $Nint(A)$. That is, $Nint(A)$ is the largest nano-open subset of A.
(ii)The nano closure of A is defined as the intersection of all nano-closed sets containing A and is denoted by $Ncl(A)$. That is, $Ncl(A)$ is the smallest nano-closed set containing A.

Definition 2.5. [5] A space U is called nano-T_0 (or N-T_0) space for $x, y \in U$ and $x \neq y$, there exists a nano-open set G such that $x \in G$ and $y \notin G$.

Definition 2.6. [5] A space U is called nano semi-T_0 (or NS-T_0) space for $x, y \in U$ and $x \neq y$, there exists a nano semi-open set G such that $x \in G$ and $y \notin G$.

Definition 2.7. [5] A space U is called nano pre-T_0 (or NP-T_0) space for $x, y \in U$ and $x \neq y$, there exists a nano pre-open set G such that $x \in G$ and $y \notin G$.

Definition 2.8. [5] A space U is called nano-T_1 (or N-T_1) space for $x, y \in U$ and $x \neq y$, there exists a nano-open sets G and H such that $x \in G$, $y \notin G$ and $y \in H$, $x \notin H$.

Definition 2.9. [5] A space U is called nano semi-T_1 (or NS-T_1) space for $x, y \in U$ and $x \neq y$, there exists a nano semi-open sets G and H such that $x \in G$, $y \notin G$ and $y \in H$, $x \notin H$.

Definition 2.10. [5] A space U is called nano pre-T_1 (or NP-T_1) space for $x, y \in U$ and $x \neq y$, there exists a nano pre-open sets G and H such that $x \in G$, $y \notin G$ and $y \in H$, $x \notin H$.

Definition 2.11. [5] A space U is called nano-T_2 (or N-T_2) space for $x, y \in U$ and $x \neq y$, there exists disjoint nano-open sets G and H such that $x \in G$ and $y \in H$.

Definition 2.12. [5] A space U is called nano semi-T_2 (or NS-T_2) space for $x, y \in U$ and $x \neq y$, there exists disjoint NS-open sets G and H such that $x \in G$ and $y \in H$.

Definition 2.13. [5] A space U is called nano pre-T_2 (or NP-T_2) space for $x, y \in U$ and $x \neq y$, there exists disjoint NP-open sets G and H such that $x \in G$ and $y \in H$.

Definition 2.14. [6] A subset $M_x \subset U$ is called a nano semi pre-neighbourhood ($N\beta$-nhd) of a point $x \in U$ iff there exists a $A \in N\beta O(U, X)$ such that $x \in A \subset M_x$ and a point x is called $N\beta$-nhd point of the set A.

3 $\mathcal{N}s\widehat{g}$-T_i spaces

Next we have defined the $\mathcal{N}s\widehat{g}$-T_i spaces where $i = 0, 1, 2$ and proved some of its basic results

Definition 3.1. A space U is called

(i) $\mathcal{N}s\widehat{g}$-T_0 if for each pair of distinct points x and y in U, there exist $\mathcal{N}s\widehat{g}$-open sets G, such that $x \in G$ and $y \notin G$.

(ii) $\mathcal{N}s\widehat{g}$-T_1 if for each pair of distinct points x and y in U, there exist $\mathcal{N}s\widehat{g}$-open sets G and H containing x and y, respectively, such that $y \notin G$ and $x \notin H$.

(iii) $\mathcal{N}s\widehat{g}$-T_2 if for each pair of distinct points x and y in U, there exist $G \in \mathcal{N}s\widehat{g}O(U, x)$ and $H \in \mathcal{N}s\widehat{g}O(U, y)$ such that $G \cap H = \emptyset$.

Definition 3.2. A nano topological space $(U, \tau_R(X))$ is $\mathcal{N}s\widehat{g}$-symmetric if for x and y in U, $x \in \mathcal{N}s\widehat{g}cl(\{y\})$ implies $y \in \mathcal{N}s\widehat{g}cl(\{x\})$.

Theorem 3.3. *Let U be the nano topological space then every $\mathcal{N}s\widehat{g}$-T_0 space is Nsg (resp. NS, NP, Ng, Ngs, Ng^*, Ng^*s)-T_0 space.*

Proof: Let U be $\mathcal{N}s\widehat{g}$-T_0-space and x and y be two distinct points of U, as U is $\mathcal{N}s\widehat{g}$-T_0 there exists nano-open set G such that $x \in G$ and $y \notin G$, since every nano-open set is Nsg (resp. NS, NP, Ng, Ngs, Ng^*, Ng^*s)-open and hence G is Nsg (resp. NS, NP, Ng, Ngs, Ng^*, Ng^*s)-T_0-open set such that $x \in G$ and $y \notin G \Rightarrow U$ is Nsg (resp. NS, NP, Ng, Ngs, Ng^*, Ng^*s)-T_0 space.
But the converse of the theorem need not be true in general.

Example 3.4. Let $U = \{a, b, c, d\}$, $UR = \{\{a\}, \{b, d\}, \{c\}\}$, $X = \{a, b\}$ and$\tau_R(X) = \{U, \phi, \{a\}, \{a, b, d\}, \{b, d\}\}$ be a nano topology on U.

(i) Let $x = \{a, c\}$ and $y = \{c\}$ then it is Nsg-T_0 (resp. NS, Ng, Ngs, Ng^*, Ng^*s-T_0 space but not $\mathcal{N}s\widehat{g}$-T_0 space.

(ii) Let $x = \{b\}$ and $y = \{c\}$ then it is NP-T_0 space but not $\mathcal{N}s\widehat{g}$-T_0 space.

Lemma 3.5. *In the $\mathcal{N}s\widehat{g}$-T_2 space, the $\mathcal{N}s\widehat{g}$-closure of every $\mathcal{N}s\widehat{g}$-open set is $\mathcal{N}s\widehat{g}$-open.*

Proof Every nano-regular open set is nano-open and every nano-open set is $\mathcal{N}s\widehat{g}$-open. by Proposition (i). Thus, every nano regular closed set is $\mathcal{N}s\widehat{g}$-closed. Now let A be any $\mathcal{N}s\widehat{g}$-open set in $\mathcal{N}s\widehat{g}$-T_2 space. There exists a $\mathcal{N}s\widehat{g}$-open set G such that $G \Rightarrow A \Rightarrow Ncl(G)$. Hence, we have $G \Rightarrow \mathcal{N}s\widehat{g}cl(G) \Rightarrow \mathcal{N}s\widehat{g}cl(A) \Rightarrow \mathcal{N}s\widehat{g}cl(Ncl(G)) = Ncl(G)$ since $Ncl(G)$ is nano regular closed. Therefore, $\mathcal{N}s\widehat{g}cl(A)$ is $\mathcal{N}s\widehat{g}$-open.

Theorem 3.6. *A topological space $(U, \tau_R(X))$ is $\mathcal{N}s\widehat{g}$-T_0 if and only if for each pair of distinct points x, y of U, $\mathcal{N}s\widehat{g}cl(\{x\}) \neq \mathcal{N}s\widehat{g}cl(\{y\})$.*

Proof Sufficiency. Suppose that x, $y \in U$, $x \neq y$ and $\mathcal{N}s\widehat{g}cl(\{x\}) \neq \mathcal{N}s\widehat{g}cl(\{y\})$. Let $z \in U$ such that $z \in \mathcal{N}s\widehat{g}cl(\{x\})$ but $z \notin \mathcal{N}s\widehat{g}cl(\{y\})$. We claim that $x \notin \mathcal{N}s\widehat{g}cl(\{y\})$. For, if $x \in \mathcal{N}s\widehat{g}cl(\{y\})$ then $\mathcal{N}s\widehat{g}cl(\{x\}) \subset \mathcal{N}s\widehat{g}cl(\{y\})$. This contradicts the fact that $z \notin \mathcal{N}s\widehat{g}cl(\{y\})$. Consequently x belongs to the $\mathcal{N}s\widehat{g}$-open set $[\mathcal{N}s\widehat{g}cl(\{y\})]^c$ to which y does not belong.
Necessity. Let $(U, \tau_R(X))$ be an $\mathcal{N}s\widehat{g}$-T_0 space and x, y be any two distinct points of U. There exists an $\mathcal{N}s\widehat{g}$-open set G containing x or y, say x but not y. Then G^c is an $\mathcal{N}s\widehat{g}$-closed set which $x \notin G^c$ and $y \in G^c$. Since $\mathcal{N}s\widehat{g}cl(\{y\})$ is the smallest $\mathcal{N}s\widehat{g}$-closed set containing y, $\mathcal{N}s\widehat{g}cl(\{y\}) \subset G^c$, and therefore $x \notin \mathcal{N}s\widehat{g}cl(\{y\})$. Hence $\mathcal{N}s\widehat{g}cl(\{x\}) \neq \mathcal{N}s\widehat{g}cl(\{y\})$.

Theorem 3.7. *A topological space $(U, \tau_R(X))$ is $\mathcal{N}s\widehat{g}$-T_1 if and only if the singletons are $\mathcal{N}s\widehat{g}$-closed sets.*

Proof Let $(U, \tau_R(X))$ be $\mathcal{N}s\widehat{g}$-T_1 and x any point of U. Suppose $y \in \{x\}^c$. Then $x \neq y$ and so there exists an $\mathcal{N}s\widehat{g}$-open set G_y such that $y \in G_y$ but $x \notin G_y$. Consequently $y \in G_y \subset \{x\}^c$ i.e., $\{x\}^c = \cup\{G_y \backslash y \in \{x\}^c\}$ which is $\mathcal{N}s\widehat{g}$-open.
Conversely, suppose $\{p\}$ is $\mathcal{N}s\widehat{g}$-closed for every $p \in U$. Let $x, y \in U$ with $x \neq y$. Now $x \neq y$ implies $y \in \{x\}^c$. Hence $\{x\}^c$ is an $\mathcal{N}s\widehat{g}$-open set containing y but not x. Similarly $\{y\}^c$ is an $\mathcal{N}s\widehat{g}$-open set containing x but not y. Accordingly U is an $\mathcal{N}s\widehat{g}$-T_1 space.

Lemma 3.8. *If a nano topological space $(U, \tau_R(X))$ is an $\mathcal{N}s\widehat{g}$-T_1 space, then it is $\mathcal{N}s\widehat{g}$-symmetric.*

Theorem 3.9. *For a nano topological space $(U, \tau_R(X))$ the following are equivalent*

(i) $(U, \tau_R(X))$ *is $\mathcal{N}s\widehat{g}$-symmetric and $\mathcal{N}s\widehat{g}$-T_0*

(ii) $(U, \tau_R(X))$ *is $\mathcal{N}s\widehat{g}$-T_1.*

Proof (1) $\rightarrow$ (2) It is obvious. (2) $\rightarrow$ (1) Let $x \neq y$ and by $\mathcal{N}s\widehat{g}$-T_0, we may assume that $x \in G_1 \subset \{y\}^c$ for some $G_1 \in \mathcal{N}s\widehat{g}O(U, \tau_R(X))$. Then $x \notin \mathcal{N}s\widehat{g}cl(\{y\})$ and hence $y \notin \mathcal{N}s\widehat{g}cl(\{x\})$. There exists a $G_\in \in \mathcal{N}s\widehat{g}O(U, \tau_R(X))$ such that $y \in G_\in \subset \{x\}^c$. Hence $(U, \tau_R(X))$ is an $\mathcal{N}s\widehat{g}$-T_1 space.

Theorem 3.10. *Let U be an arbitrary space, R an equivalence relation in U and $f : U \rightarrow U/R$ the identification map. If $R \subset U \times U$ is $\mathcal{N}s\widehat{g}$-closed in $U \times U$ and p is an always $\mathcal{N}s\widehat{g}$-open map, then UR is $\mathcal{N}s\widehat{g}$-T_2.*

Proof Let $p(x)$, $p(y)$ be distinct members of UR. Since x and y are not related, $R \subset U \times U$ is $\mathcal{N}s\widehat{g}$-closed in $U \times U$. There are $\mathcal{N}s\widehat{g}$-open sets G and H such that $x \in G$, $y \in H$ and $G \times H \subset R^c$. Thus $f(G)$, $f(H)$ are disjoint and also $\mathcal{N}s\widehat{g}$-open in UR since f is always $\mathcal{N}s\widehat{g}$-open.

Theorem 3.11. *A space U is $\mathcal{N}s\widehat{g}$-T_2 if and only if for any pair of distinct points x, y of U there exist $\mathcal{N}s\widehat{g}$-open sets G and H such that $x \in G$ and $y \in H$ and $\mathcal{N}s\widehat{g}cl(G) \cap \mathcal{N}s\widehat{g}cl(H) = \emptyset$.*

Proof Necessity. Suppose that U is $\mathcal{N}s\widehat{g}$-T_2. Let x and y be distinct points of x. There exist $\mathcal{N}s\widehat{g}$-open sets G and H such that $x \in G, y \in H$ and $G \cap H = \emptyset$. Hence $\mathcal{N}s\widehat{g}cl(G) \cap \mathcal{N}s\widehat{g}cl(H) = \emptyset$ and by above Lemma, $\mathcal{N}s\widehat{g}cl(G)$ is $\mathcal{N}s\widehat{g}$-open. Therefore, we obtain $\mathcal{N}s\widehat{g}cl(G) \cap \mathcal{N}s\widehat{g}cl(H) = \emptyset$. Sufficiency. This is obvious.

4 $\mathcal{N}s\widehat{g}$-REGULAR AND $\mathcal{N}s\widehat{g}$-NORMAL SPACES

In this section we introduce and investigated the concept of $\mathcal{N}s\widehat{g}$-regular space and its properties also we have proved results which satisfies the definition.

Definition 4.1. A space $(U, \tau_R(X))$ is said to be $\mathcal{N}s\widehat{g}$-regular if for every $\mathcal{N}s\widehat{g}$-closed set F and each point $x \notin F$, there exist disjoint nano open sets G and H such that $F \Rightarrow G$ and $x \in H$.

Remark 4.2. Let $(U, \tau_R(X))$ be a nano topological space. Then every $\mathcal{N}s\widehat{g}$-regular is Nsg (resp. NS, NP, Ng, Ngs, Ng^*, Ng^*s) space.

Example 4.3. From Example 3.3, let $F = \{b, d\}$ and $x = \{c\}$ then it is Nsg (resp. NS, NP, Ng, Ngs, Ng^*, Ng^*s) space but not $\mathcal{N}s\widehat{g}$-regular space.

Theorem 4.4. *Let $(U, \tau_R(X))$ be a nano topological space. Then the following statements are equivalent*
(i) $(U, \tau_R(X))$ is a $\mathcal{N}s\widehat{g}$-regular space.
(ii) For each $x \in U$ and $\mathcal{N}s\widehat{g}$-neighbourhood W of x there exists an nano-open neighbourhood H of x such that $Ncl(H) \subseteq W$.

Proof (i) $\Rightarrow$ (ii). Let W be any $\mathcal{N}s\widehat{g}$-neighbourhood of x. Then there exist a $\mathcal{N}s\widehat{g}$-open set G such that $x \in G \Rightarrow W$. Since G^c is $\mathcal{N}s\widehat{g}$-closed and $x \notin G^c$, by hypothesis there exist nano-open sets G and H such that $G^c \Rightarrow G$, $x \in H$ and $G \cap H = \emptyset$ and so $H \Rightarrow G^c$. Now, $Ncl(H) \Rightarrow Ncl(G^c) = G^c$ and $G^c \Rightarrow G$ implies $G^c \Rightarrow G \Rightarrow W$. Therefore $Ncl(H) \Rightarrow W$.
(ii) $\Rightarrow$ (i). Let F be any $\mathcal{N}s\widehat{g}$-closed set and $x \notin F$. Then $x \in F^c$ and F^c is $\mathcal{N}s\widehat{g}$-open and so F^c is an $\mathcal{N}s\widehat{g}$-neighbourhood of x. By hypothesis, there exists an nano-open neighbourhood H of x such that $x \in H$ and $Ncl(H) \Rightarrow F^c$, which implies $F \Rightarrow (Ncl(H))^c$. Then $(Ncl(H))^c$ is an nano-open set containing F and $H \cap (Ncl(H))^c = \emptyset$. Therefore, U is $\mathcal{N}s\widehat{g}$-regular.

Theorem 4.5. *For a space $(U, \tau_R(X))$ the following are equivalent: (i) $(U, \tau_R(X))$ is nano normal.*
(ii) For every pair of disjoint nano closed sets A and B, there exist $\mathcal{N}s\widehat{g}$- open sets G and H such that $A \Rightarrow G$, $B \Rightarrow H$ and $G \cap H = \emptyset$.

Proof (i) $\Rightarrow$ (ii). Let A and B be disjoint nano closed subsets of $(U, \tau_R(X))$. By hypothesis, there exist disjoint nano open sets (and hence $\mathcal{N}s\widehat{g}$-open sets) G and H such that $A \Rightarrow G$ and $B \Rightarrow H$.
(ii) $\Rightarrow$ (i). Let A and B be nano closed subsets of $(U, \tau_R(X))$. Then by assumption, $A \Rightarrow G, B \Rightarrow H$ and $G \cap H = \emptyset$, where G and H are disjoint $\mathcal{N}s\widehat{g}$-open sets. Since A and B are Nsg-closed, $A \Rightarrow Nint(G)$ and $B \Rightarrow Nint(H)$. Further, $Nint(G) \cap Nint(H) = Nint(G \cap H) = \emptyset$.

Lemma 4.6. *A $gT\mathcal{N}s\widehat{g}$-space $(U, \tau_R(X))$ is symmetric if and only if $\{x\}$ is $\mathcal{N}s\widehat{g}$-closed in $(U, \tau_R(X))$ for each point x of $(U, \tau_R(X))$.*

Theorem 4.7. *A nano topological space $(U, \tau_R(X))$ is $\mathcal{N}s\widehat{g}$-regular if and only if for each $\mathcal{N}s\widehat{g}$-closed set F of $(U, \tau_R(X))$ and each $x \in F^c$ there exist nano open sets G and H of $(U, \tau_R(X))$ such that $x \in G, F \Rightarrow H$ and $Ncl(G) \cap Ncl(H) = \emptyset$.*

Proof Let F be a $\mathcal{N}s\widehat{g}$-closed set of $(U, \tau_R(X))$ and $x \notin F$. Then there exist nano open sets G_0 and H of $(U, \tau_R(X))$ such that $x \in G_0, F \Rightarrow H$ and $G_0 \cap H = \emptyset$, which implies $G_0 \cap Ncl(H) = \emptyset$. Since $Ncl(H)$ is nano closed, it is $\mathcal{N}s\widehat{g}$-closed and $x \notin Ncl(H)$. Since $(U, \tau_R(X))$ is $\mathcal{N}s\widehat{g}$-regular, there exist nano open sets G and H of $(U, \tau_R(X))$ such that $x \in G, Ncl(H) \Rightarrow H$ and $G \cap H = \emptyset$, which implies $Ncl(G) \cap H = \emptyset$. Let $G = G_0 \cap G$, then G and H are nano open sets of $(U, \tau_R(X))$ such that $x \in G, F \Rightarrow H$ and $Ncl(G) \cap Ncl(H) = \emptyset$.
Converse part is trivial.

Corollary 4.8. *If a nano topological space $(U, \tau_R(X))$ is $\mathcal{N}s\widehat{g}$-regular, symmetric and $gT\mathcal{N}s\widehat{g}$-space, then it is Urysohn.*

Proof Let x and y be any two distinct points of $(U, \tau_R(X))$. Since $(U, \tau_R(X))$ is symmetric and $gT\mathcal{N}s\widehat{g}$-space, x is $\mathcal{N}s\widehat{g}$-closed by Lemma 4.5. Therefore, by Theorem 4.6, there exist nano open sets G and H such that $x \in G, y \in H$ and $Ncl(G) \cap Ncl(H) = \emptyset$.

Theorem 4.9. *Let $(U, \tau_R(X))$ be a nano topological space. Then the following statements are equivalent:*
(i) $(U, \tau_R(X))$ is $\mathcal{N}s\hat{g}$-regular.
(ii) For each point $x \in U$ and for each $\mathcal{N}s\hat{g}$-neighbourhood W of x, there exists an nano open neighbourhood H of x such that $Ncl(H) \Rightarrow W$.
(iii) For each point $x \in U$ and for each $\mathcal{N}s\hat{g}$-closed set F not containing x, there exists an nano open neighbourhood H of x such that $Ncl(H) \cap F = \emptyset$.

Proof (i) $\Rightarrow$ (ii). It is obvious from Theorem 4.3.
(ii) $\Rightarrow$ (iii). Let $x \in U$ and F be a $\mathcal{N}s\hat{g}$-closed set such that $x \notin F$. Then F^c is a $\mathcal{N}s\hat{g}$-neighbourhood of x and by hypothesis, there exists an nano open neighbourhood H of x such that $Ncl(H) \Rightarrow F^c$ and hence $Ncl(H) \cap F = \emptyset$.
(iii) $\Rightarrow$ (ii). Let $x \in U$ and W be a $\mathcal{N}s\hat{g}$-neighbourhood of x. Then there exists a $\mathcal{N}s\hat{g}$-open set G such that $x \in G \Rightarrow W$. Since G^c is $\mathcal{N}s\hat{g}$-closed and $x \notin G^c$, by hypothesis there exists an nano open neighbourhood H of x such that $Ncl(H) \cap G^c = \emptyset$. Therefore, $Ncl(H) \Rightarrow G \Rightarrow W$.

Theorem 4.10. *The following are equivalent for a space $(U, \tau_R(X))$.*
(i) $(U, \tau_R(X))$ is $\mathcal{N}s\hat{g}$-regular.
(ii) $Ncl \Rightarrow (A) = \mathcal{N}s\hat{g}cl(A)$ for each subset A of $(U, \tau_R(X))$.
(iii) $Ncl \Rightarrow (A) = A$ for each $\mathcal{N}s\hat{g}$-closed set A.

Proof (i) $\Rightarrow$ (ii). For any subset A of $(U, \tau_R(X))$, we have always $A \Rightarrow \mathcal{N}s\hat{g}cl(A) \Rightarrow Ncl \Rightarrow (A)$. Let $x \in (\mathcal{N}s\hat{g}cl(A))^c$. Then there exists a $\mathcal{N}s\hat{g}$-closed set F such that $x \in F^c$ and $A \Rightarrow F$. By assumption, there exist disjoint nano-open sets G and H such that $x \in G$ and $F \Rightarrow H$. Now, $x \in G \Rightarrow Ncl(G) \Rightarrow H^c \Rightarrow F^c \Rightarrow A^c$ and therefore $Ncl(G) \cap A = \emptyset$. Thus, $x \in (Ncl \Rightarrow (A))^c$ and hence $Ncl \Rightarrow (A) = \mathcal{N}s\hat{g}cl(A)$.
(ii) $\Rightarrow$ (iii). It is trivial.
(iii) $\Rightarrow$ (i). Let F be any $\mathcal{N}s\hat{g}$-closed set and $x \in F^c$. Since F is $\mathcal{N}s\hat{g}$- closed, by assumption $x \in (Ncl \Rightarrow (F))^c$ and so there exists an nano-open set G such that $x \in G$ and $Ncl(G) \cap F = \emptyset$. Then $F \Rightarrow (Ncl(G))^c$. Let $H = (Ncl(G))^c$. Then H is an nano-open set such that $F \Rightarrow H$. Also, the sets G and H are disjoint and hence $(U, \tau_R(X))$ are $\mathcal{N}s\hat{g}$-regular.

Theorem 4.11. *If $(U, \tau_R(X))$ is a $\mathcal{N}s\hat{g}$-regular space and $f : (U, \tau_R(X)) \Rightarrow (V, \tau_{R'}(Y))$ is bijective, nano pre-sg-open, $\mathcal{N}s\hat{g}$-continuous and nano-open, then $(V, \tau_{R'}(Y))$ is $\mathcal{N}s\hat{g}$-regular.*

Proof Let F be any $\mathcal{N}s\hat{g}$-closed subset of $(V, \tau_{R'}(Y))$ and $y \notin F$. Since the function f is $\mathcal{N}s\hat{g}$-irresolute, we have $f^{-1}(F)$ is $\mathcal{N}s\hat{g}$-closed in $(U, \tau_R(X))$. Since f is bijective, let $f(x) = y$, then $x \notin f^{-1}(F)$. By hypothesis, there exist disjoint nano-open sets G and H such that $x \in G$ and $f^{-1}(F) \Rightarrow H$. Since f is nano-open and bijective, we have $y \in f(G)$, $F \Rightarrow f(H)$ and $f(G) \cap f(H) = \emptyset$. This shows that the space $(V, \tau_{R'}(Y))$ is also $\mathcal{N}s\hat{g}$-regular.

Theorem 4.12. *If $f : (U, \tau_R(X)) \Rightarrow (V, \tau_{R'}(Y))$ is Nsg-irresolute $\mathcal{N}s\hat{g}$-closed continuous injection and $(V, \tau_{R'}(Y))$ is $\mathcal{N}s\hat{g}$-regular, then $(U, \tau_R(X))$ is $\mathcal{N}s\hat{g}$-regular.*

Proof Let F be any $\mathcal{N}s\hat{g}$-closed set of $(U, \tau_R(X))$ and $x \notin F$. Since f is Nsg-irresolute $\mathcal{N}s\hat{g}$-closed, $f(F)$ is $\mathcal{N}s\hat{g}$-closed in $(V, \tau_{R'}(Y))$ and $f(x) \notin f(F)$. Since $(V, \tau_{R'}(Y))$ is $\mathcal{N}s\hat{g}$-regular and so there exist disjoint nano-open sets G and H in $(V, \tau_{R'}(Y))$ such that $f(x) \in G$ and $f(F) \Rightarrow H$. i.e., $x \in f^{-1}(G)$, $F \Rightarrow f^{-1}(H)$ and $f^{-1}(G) \cap f^{-1}(H) = \emptyset$. Therefore, $(U, \tau_R(X))$ is $\mathcal{N}s\hat{g}$-regular.

Theorem 4.13. *If $f : (U, \tau_R(X)) \Rightarrow (V, \tau_{R'}(Y))$ is nano-weakly continuous $\mathcal{N}s\hat{g}$-closed injection and $(V, \tau_{R'}(Y))$ is $\mathcal{N}s\hat{g}$-regular, then $(U, \tau_R(X))$ is nano-regular.*

Proof Let F be any nano-closed set of $(U, \tau_R(X))$ and $x \notin F$. Since f is $\mathcal{N}s\hat{g}$-closed, $f(F)$ is $\mathcal{N}s\hat{g}$-closed in $(V, \tau_{R'}(Y))$ and $f(x) \notin f(F)$. Since $(V, \tau_{R'}(Y))$ is $\mathcal{N}s\hat{g}$-regular by Theorem 4.6 there exist nano-open sets G and H such that $f(x) \in G, f(F) \Rightarrow H$ and $Ncl(G) \cap Ncl(H) = \emptyset$. Since f is nano-weakly continuous, $x \in f^{-1}(G) \Rightarrow Nint(f^{-1}(Ncl(G)))$, $F \Rightarrow f^{-1}(H) \Rightarrow Nint(f^{-1}(Ncl(H)))$ and $Nint(f^{-1}(Ncl(G))) \cap Nint(f^{-1}(Ncl(H))) = \emptyset$. Therefore, $(U, \tau_R(X))$ is nano-regular. We conclude this section with the introduction of $\mathcal{N}s\hat{g}$-normal space in topological spaces.

Definition 4.14. A topological space $(U, \tau_R(X))$ is said to be $\mathcal{N}s\widehat{g}$-normal if for any pair of disjoint $\mathcal{N}s\widehat{g}$-closed sets A and B, there exist disjoint nano-open sets G and H such that $A \Rightarrow G$ and $B \Rightarrow H$.

We characterize $\mathcal{N}s\widehat{g}$-normal space.

Theorem 4.15. *Let $(U, \tau_R(X))$ be a nano topological space. Then the following statements are equivalent:*
(i) $(U, \tau_R(X))$ is $\mathcal{N}s\widehat{g}$-normal.
(ii) For each $\mathcal{N}s\widehat{g}$-closed set F and for each $\mathcal{N}s\widehat{g}$-open set G containing F, there exists an nano-open set H containing F such that $Ncl(H) \Rightarrow G$.
(iii) For each pair of disjoint $\mathcal{N}s\widehat{g}$-closed sets A and B in $(U, \tau_R(X))$, there exists an nano-open set G containing A such that $Ncl(G) \cap B = \emptyset$.
(iv) For each pair of disjoint $\mathcal{N}s\widehat{g}$-closed sets A and B in $(U, \tau_R(X))$, there exist nano-open sets G containing A and H containing B such that $Ncl(G) \cap Ncl(H) = \emptyset$.

Proof (i) $\Rightarrow$ (ii). Let F be a $\mathcal{N}s\widehat{g}$-closed set and G be a $\mathcal{N}s\widehat{g}$-open set such that $F \Rightarrow G$. Then $F \cap G^c = \emptyset$. By assumption, there exist nano-open sets H and W such that $F \Rightarrow H, G^c \Rightarrow W$ and $H \cap W = \emptyset$, which implies $Ncl(H) \cap W = \emptyset$. Now, $Ncl(H) \cap G^c \Rightarrow Ncl(H) \cap W = \emptyset$ and so $Ncl(H) \Rightarrow G$.
(ii) $\Rightarrow$ (iii). Let A and B be disjoint $\mathcal{N}s\widehat{g}$-closed sets of $(U, \tau_R(X))$. Since $A \cap B = \emptyset, A \Rightarrow B^c$ and B^c is $\mathcal{N}s\widehat{g}$-open. By assumption, there exists an nano-open set G containing A such that $Ncl(G) \Rightarrow B^c$ and so $Ncl(G) \cap B = \emptyset$.
(iii) $\Rightarrow$ (iv). Let A and B be any two disjoint $\mathcal{N}s\widehat{g}$-closed sets of $(U, \tau_R(X))$. Then by assumption, there exists an nano-open set G containing A such that $Ncl(G) \cap B = \emptyset$. Since $Ncl(G)$ is nano-closed, it is $\mathcal{N}s\widehat{g}$-closed and so B and $Ncl(G)$ are disjoint $\mathcal{N}s\widehat{g}$-closed sets in $(U, \tau_R(X))$. Therefore again by assumption, there exists an nano-open set H containing B such that $Ncl(H) \cap Ncl(G) = \emptyset$.
(iv) $\Rightarrow$ (i). Let A and B be any two disjoint $\mathcal{N}s\widehat{g}$-closed sets of $(U, \tau_R(X))$. By assumption, there exist nano-open sets G containing A and H containing B such that $Ncl(G) \cap Ncl(H) = \emptyset$, we have $G \cap H = \emptyset$ and thus $(U, \tau_R(X))$ is $\mathcal{N}s\widehat{g}$- normal.

Theorem 4.16. *If $f : (U, \tau_R(X)) \Rightarrow (V, \tau_{R'}(Y))$ is bijective, nano-pre-sg-open, $\mathcal{N}s\widehat{g}$-continuous and nano-open and $(U, \tau_R(X))$ is $\mathcal{N}s\widehat{g}$-normal, then $(V, \tau_{R'}(Y))$ is $\mathcal{N}s\widehat{g}$-normal.*

Proof Let A and B be any disjoint $\mathcal{N}s\widehat{g}$-closed sets of $(V, \tau_{R'}(Y))$. The function f is $\mathcal{N}s\widehat{g}$-irresolute and so $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint $\mathcal{N}s\widehat{g}$- closed sets of $(U, \tau_R(X))$. Since $(U, \tau_R(X))$ is $\mathcal{N}s\widehat{g}$-normal, there exist disjoint nano-open sets G and H such that $f^{-1}(A) \Rightarrow G$ and $f^{-1}(B) \Rightarrow H$. Since f is nano-open and bijective, we have $f(G)$ and $f(H)$ are nano-open in $(V, \tau_{R'}(Y))$ such that $A \Rightarrow f(G), B \Rightarrow f(H)$ and $f(G) \cap f(H) = \emptyset$. Therefore, $(V, \tau_{R'}(Y))$ is $\mathcal{N}s\widehat{g}$-normal.

Theorem 4.17. *If $f : (U, \tau_R(X)) \Rightarrow (V, \tau_{R'}(Y))$ is Nsg-irresolute $\mathcal{N}s\widehat{g}$-closed continuous injection and $(V, \tau_{R'}(Y))$ is $\mathcal{N}s\widehat{g}$-normal, then $(U, \tau_R(X))$ is $\mathcal{N}s\widehat{g}$-normal.*

Proof Let A and B be any disjoint $\mathcal{N}s\widehat{g}$-closed subsets of $(U, \tau_R(X))$. Since f is Nsg-irresolute $\mathcal{N}s\widehat{g}$-closed, $f(A)$ and $f(B)$ are disjoint $\mathcal{N}s\widehat{g}$-closed sets of $(V, \tau_{R'}(Y))$. Since $(V, \tau_{R'}(Y))$ is $\mathcal{N}s\widehat{g}$-normal, there exist disjoint nano-open sets G and H such that $f(A) \Rightarrow G$ and $f(B) \Rightarrow H$. i.e., $A \Rightarrow f^{-1}(G), B \Rightarrow f^{-1}(H)$ and $f^{-1}(G) \cap f^{-1}(H) = \emptyset$. Since f is nano-continuous, $f^{-1}(G)$ and $f^{-1}(H)$ are nano-open in $(U, \tau_R(X))$, we have $(U, \tau_R(X))$ is $\mathcal{N}s\widehat{g}$-normal.

Theorem 4.18. *If $f : (U, \tau_R(X)) \Rightarrow (V, \tau_{R'}(Y))$ is weakly continuous $\mathcal{N}s\widehat{g}$-closed injection and $(V, \tau_{R'}(Y))$ is $\mathcal{N}s\widehat{g}$-normal, then $(U, \tau_R(X))$ is normal.*

Proof Let A and B be any two disjoint nano closed sets of $(U, \tau_R(X))$. Since f is injective and $\mathcal{N}s\widehat{g}$-closed, $f(A)$ and $f(B)$ are disjoint $\mathcal{N}s\widehat{g}$-closed sets of $(V, \tau_{R'}(Y))$. Since $(V, \tau_{R'}(Y))$ is $\mathcal{N}s\widehat{g}$-normal, by Theorem 4.14, there exist nano open sets G and H such that $f(A) \Rightarrow G, f(B) \Rightarrow H$ and $Ncl(G) \cap Ncl(H) = \emptyset$. Since f is nano weakly continuous. $A \Rightarrow f^{-1}(G) \Rightarrow Nint(f^{-1}(Ncl(G))), B \Rightarrow f^{-1}(H) \Rightarrow Nint(f^{-1}(Ncl(H)))$ and $Nint(f^{-1}(Ncl(G))) \cap Nint(f^{-1}(Ncl(H)))$ $\emptyset$. Therefore, $(U, \tau_R(X))$ is nano-normal.

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