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ON r-DYNAMIC COLORING OF SUBDIVISION - VERTEX JOIN OF TWO GRAPHS

Aparna V and Mohanapriya N

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Abstract Let H be a simple graph with vertex set $V(H)$ and edge set $E(H)$ which is connected, undirected and finite. For positive integers r, the proper k-coloring of the vertices of the graph H such that $|f(N(z))| \geq \min\{r, d(z)\}$ for each $z \in V(H)$ is referred to as r-dynamic coloring of a graph H. Here $N(z)$ denotes the neighborhood of the vertex z and $d(z)$ is the degree of the vertex z. The least k which permits H to have an r-dynamic coloring with k colors is called the r-dynamic chromatic number of the graph H and it is denoted as $\chi_r(H)$. The subdivision - vertex join of two graphs H_1 and H_2 denoted as $H_1 \dot{\vee} H_2$ is acquired from the sub-division graph $S(H_1)$ and H_2 by connecting each old vertex of H_1 with every vertex of H_2. In this paper we have acquired the r-dynamic chromatic number of subdivision - vertex join of path P_n with path P_m, complete graph K_m and star graph $K_{1,m}$.

1 Introduction and Preliminaries

The idea of r-dynamic coloring was put forward by Bruce Montgomery in [9]. By the word proper vertex coloring of a graph we mean a coloring where any two adjacent vertices receive distinct colors. Let $N(z)$ and $d(z)$ denotes the neighborhood set of vertex z and number of vertices adjacent to z respectively then for each positive integer r, the r-dynamic coloring of H is a proper vertex coloring f such that $|f(N(z))| \geq \min\{r, d(z)\}$, for every $z \in V(H)$ i.e. the neighbors of each vertex z acquires at least $\min\{r, d(z)\}$ distinct colors. The least k which permits H to have an r-dynamic coloring with k colors is referred to as the r-dynamic chromatic number[4, 10] of the graph H and it is denoted as $\chi_r(H)$. The 1-dynamic chromatic number is the normal chromatic number $\chi(H)$ and the 2-dynamic chromatic number is simply called the dynamic chromatic number of H and is denoted as $\chi_d(H)$. In the following papers [1, 2, 3, 6, 7, 8] the dynamic coloring of graphs has been analyzed in depth. $\chi_r(H) \geq min\{r, \Delta(H)\} + 1$ is one of the most familiar lower bound for $\chi_r(H)$ and it was put forward by Montgomery and Lai in [6].

The subdivision graph $S(H)$ of a graph H is acquired by inserting a new vertex for every edge of H. The subdivision - vertex join [5] of two graphs H_1 and H_2 denoted as $H_1 \dot{\vee} H_2$ is acquired from the sub-division graph $S(H_1)$ and H_2 by connecting each old vertex of H_1 with every vertex of H_2. Consider the graph H_1 having n vertices, t edges and H_2 having m vertices. Let the vertex set and edge set of H_1 be defined as $V(H_1) = \{u_1, u_2, \cdots, u_n\}$, $E(H_1) = \{a_1, a_2, \cdots, a_t\}$ and let the vertex set of H_2 be $V(H_2) = \{v_1, v_2, \cdots, v_m\}$ then the vertex set of $H_1 \dot{\vee} H_2$ be defined as $\{u_1, u_2, \cdots, u_n\} \cup \{a_1, a_2, \cdots, a_t\} \cup \{v_1, v_2, \cdots, v_m\}$. The star graph $K_{1,m}$ is a complete bipartite graph with $m+1$ vertices in which the single vertex belongs to one set and the remaining t vertices belongs to the other set.

2 Theorems

Theorem 2.1. *For positive integers n, m, the r- dynamic chromatic number of subdivision - vertex join of path P_n with path P_m is*

I. *When* $n \geq 4, m \geq 3$ *and* $m < n$

$$\chi_r(P_n \dot{\vee} P_m) = \begin{cases} r+2 & : \ 1 \leq r \leq 3 \\ 2r-1 & : \ 4 \leq r \leq m, m \geq 4 \\ r+m-1 & : \ m+1 \leq r \leq n+1, m \geq 3 \\ m+n & : \ r = n+2 \end{cases}$$

Proof. Let the edge set of P_n be $E(P_n) = \{a_1, a_2, \cdots, a_{n-1}\}$. Then vertex set of $P_n \dot{\vee} P_m$ is $V(P_n \dot{\vee} P_m) = \{u_1, u_2, \cdots, u_n\} \cup \{a_1, a_2, \cdots, a_{n-1}\} \cup \{v_1, v_2, \cdots, v_m\}$. The edge set of $P_n \dot{\vee} P_m$ is $E(P_n \dot{\vee} P_m) = \{u_i a_i : 1 \leq i \leq n-1\} \cup \{u_i a_{i-1} : 2 \leq i \leq n\} \cup \{v_j v_{j+1} : 1 \leq j \leq m-1\} \cup \{u_i v_j : 1 \leq i \leq n, 1 \leq j \leq m\}$. The minimum degree and maximum degree in this case is 2 and $n+2$ respectively.

Case 1 : When $1 \leq r \leq 3$.

Subcase 1 : $r = 1$

The presence of clique of order 3 gives us the fact we require at least 3 different colors. Hence the lower bound $\chi_r(P_n \dot{\vee} P_m) \geq 3$. We provide the upper bound using the mapping $f : V(P_n \dot{\vee} P_m) \to \{1, 2, 3\}$ as follows:

$f(u_i) = 1$ for all i

$f(a_i) = 1$ for $1 \leq i \leq n-1$

$$f(v_j) = \begin{cases} 2, & when\ j\ is\ odd \\ 3, & when\ j\ is\ even \end{cases}$$

This gives the upper bound $\chi_r(P_n \dot{\vee} P_m) \leq 3$ and hence $\chi_r(P_n \dot{\vee} P_m) = 3 = r+2$.

Subcase 2 : $2 \leq r \leq 3$.

Consider the vertex a_1 which is of degree 2 inorder to satisfy its 2-adjacency provide the colors 1, 2 and 3 to u_1, a_1, u_2 respectively. Now while considering the vertex u_1 for satisfying its 2-adjacency we need to provide a new color $4 = r+2$ to any v_j since neither the color 1 and 3 can be applied to v_j. Similarly when $r = 3$ for satisfying the r-adjacency condition of u_1 we need to provide the colors 4 and $5 = r+2$ to any of the two v_j's. Hence we require a minimum of $r+2$ different colors here i.e., $\chi_r(P_n \dot{\vee} P_m) \geq r+2$. The upper bound is given by the map $f : V(P_n \dot{\vee} P_m) \to \{1, 2, \cdots, r+2\}$.

$$f(u_i) = \begin{cases} 1, & when\ i\ is\ odd \\ 3, & when\ i\ is\ even \end{cases}$$

$f(a_i) = 2$ for $1 \leq i \leq n-1$

$f(v_1, v_2, \cdots, v_m) = \{2, 4, \cdots, r+2, 2, 4, \cdots, r+2, \cdots\}$

Hence $\chi_r(P_n \dot{\vee} P_m) = r+2$.

Case 2 : When $4 \leq r \leq m, m \geq 4$.

Here in this case we consider $m \geq 4$ and the case when $m = 3$ does not come under this case because in this case the value of r varies from 4 to m so it belongs to the next case. Consider the vertex u_1 and let it be assigned the color 1 also let the vertices a_1, u_2 be assigned the colors 2 and 3 respectively. Now in order to satisfy the r-adjacency condition of u_1 we provide the colors $4, \cdots, r+2$ to the vertices v_j's in order and this case ends at $r = m$. Now consider the vertex v_1 it is already adjacent to the vertex u_1, u_2 with colors 1 and 3 in order to satisfy the r-adjacency condition we need to provide the colors $r+3, \cdots, 2r-1$ to the remaining vertices of u_i since $n > m$. Hence we have the lower bound $\chi_r(P_n \dot{\vee} P_m) \geq 2r-1$. Consider the map $f : V(P_n \dot{\vee} P_m) \to \{1, 2, \cdots, 2r-1\}$ and the coloring is as below.

$f(u_1, u_2, \cdots, u_n) = \{1, 3, r+3, \cdots, 2r-1, 1, 3, r+3, \cdots, 2r-1, \cdots\}$

$f(a_i) = 2$ for $1 \leq i \leq n-1$

$f(v_1, v_2, \cdots, v_m) = \{2, 4, \cdots, r+2, 2, 4, \cdots, r+2, \cdots\}$

This gives us the upper bound as $\chi_r(P_n \dot{\vee} P_m) \leq 2r-1$ and hence $\chi_r(P_n \dot{\vee} P_m) = 2r-1$.

Case 3 : When $m+1 \leq r \leq n+1, n \geq 3$.

Let us first assign the vertices u_1, a_1, u_2 with the colors 1, 2, 3 respectively. Now the vertex u_1

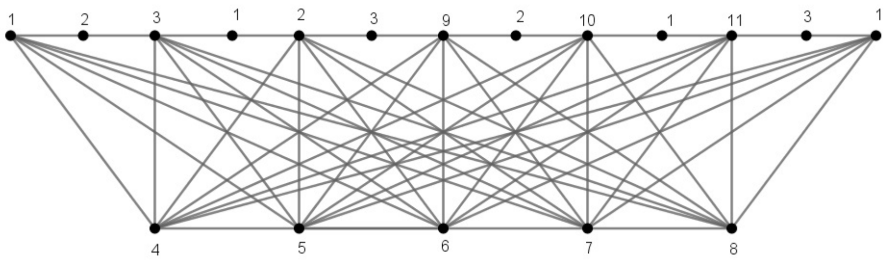

Figure 1. The 7-dynamic coloring of the graph $P_7\dot{\vee}P_5$

with degree $m+1$ needs $m+1$ different colored neighbors hence assign the colors $4,\cdots,m+3$ colors to $v_1,v_2,\cdots,v_m$. Now for satisfying the r-adjacency of the vertices v_j we provide the colors $m+4,\cdots,r+m-1$ to the remaining u_i's, $i\geq 3$. Thus we require a minimum of at least $r+m-1$ colors in this case hence $\chi_r(P_n\dot{\vee}P_m)\geq r+m-1$. The coloring is given below using the map $f:V(P_n\dot{\vee}P_m)\to\{1,2,\cdots,r+m-1\}$.
When $m=3$ and $r=4$ the coloring is:
$f(u_1,u_2,\cdots,u_n)=\{1,3,2,1,3,2,\cdots\}$
$f(v_1,v_2,v_3)=\{4,5,6\}$ and for $\{a_i:1\leq i\leq n-1\}$ provide suitable color from the set of colors $\{1,2,3\}$ so that each a_i satisfies 2-adjacency condition.
For all the remaining case the coloring is as below.
$f(u_1,u_2,\cdots,u_n)=\{1,3,m+4,\cdots,r+m-1,1,3,m+4,\cdots,r+m-1,\cdots\}$
For $\{a_i:1\leq i\leq n-1\}$ provide the coloring as said for $m=3$ and $r=4$.
$f(v_1,v_2,\cdots,v_m)=\{4,5,\cdots,m+3\}$
Thus $\chi_r(P_n\dot{\vee}P_m)=r+m-1$.

Case 4 : When $r=\Delta=n+2$.
By the case $r=n+1$ the r-adjacencies of all the vertices will be satisfied and we no longer require any new colors other than the $m+n$ colors used in the case $r=n+1$. The coloring in this case is as below.
$f(u_1,u_2,\cdots,u_n)=\{1,3,m+4,\cdots,m+n\}$
For the $\{a_i:1\leq i\leq n-1\}$ provide suitable color from the set of colors $\{1,2,3\}$ so that each a_i satisfies 2-adjacency condition.
$f(v_1,v_2,\cdots,v_m)=\{4,5,\cdots,m+3\}$
Hence $\chi_r(P_n\dot{\vee}P_m)=m+n$. $\square$

II. When $n\geq 4$ and $m\geq n$

$$\chi_r(P_n\dot{\vee}P_m)=\begin{cases} r+2 & : \quad 1\leq r\leq 3\\ 2r-1 & : \quad 4\leq r\leq n+1, m>n \; and \; 4\leq r\leq n, m=n\\ r+n & : \quad n+2\leq r\leq m, m\geq n+2\\ m+n & : \quad r=m+1, m+2, m\geq n\end{cases}$$

Proof. The maximum and minimum degrees in this case are $m+2$ and 2 respectively. The cases when $1\leq r\leq 3$ is same as the one given in the earlier part of the theorem.
Case 2 : When $4\leq r\leq n+1, m>n$ and $4\leq r\leq n, m=n$.
When $m=n$ the case ends at $r=n$ and the coloring for $r=n+1$ goes to Case 4. Let us first assign the vertices u_1,a_1,u_2 with the colors 1, 2, 3 respectively. Now in order to satisfy the r-adjacency condition of u_1 we provide the colors $4,\cdots,r+2$ to the vertices v_j's in order. Also while considering the vertex v_1 it is already adjacent to the vertex u_1, u_2 with colors 1 and 3 in order to satisfy the r-adjacency condition we need to provide the colors $r+3,\cdots,2r-1$ to the remaining vertices of u_i and this case ends at $r=n+1$ since the degree of v_1 is $n+1$. The coloring in this case is same as the one given in Case 2 of first part of the theorem. Thus $\chi_r(P_n\dot{\vee}P_m)=2r-1$.

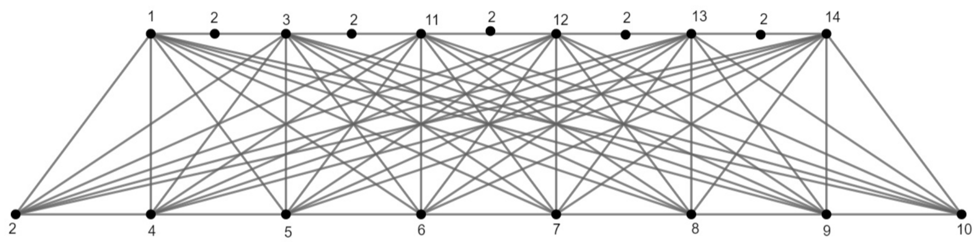

Figure 2. The 8-dynamic coloring of the graph $P_6\dot{\vee}P_8$

Case 3 : When $n+2 \leq r \leq m, m \geq n+2$.
The condition $m \geq n+2$ is necessary otherwise it will not be well-defined as in the case of $m = n, n+1$. For $m = n$, $r = n+2 = \Delta$ and $m = n+1$, $r = n+2 = m+1$ is provided in Case 4. Now for the remaining $m \geq n+2$, in order to satisfy the r-adjacency condition of u_1 and v_i we provide the colors $4, \cdots, r+2$ to the vertices v_j's in order and $1, 3, r+3, \cdots, r+n$ to the vertices u_i in order. Hence we have the lower bound $\chi_r(P_n\dot{\vee}P_m) \geq r+n$. The coloring is this case is defined by the mapping $f : V(P_n\dot{\vee}P_m) \to \{1, 2, \cdots, r+n\}$.
$f(u_1, u_2, \cdots, u_n) = \{1, 3, r+3, \cdots, r+n, 1, 3, r+3, \cdots, r+n, \cdots\}$
$f(a_i) = 2$
$f(v_1, v_2, \cdots, v_m) = \{2, 4, 5, \cdots, r+2, 2, 4, 5, \cdots, r+2, \cdots\}$
Thus $\chi_r(P_n\dot{\vee}P_m) = r+n$.

Case 4 : When $r = m+1, m+2, m \geq n$.
When $r = m+1$ for satisfying the $m+1$-adjacency of the vertex u_1 we first provide the colors $4, \cdots, m+3$ to the m vertices of P_m with the assumption that u_1, a_1, u_2 are assigned the color $1, 2, 3$. Now for satisfying the r-adjacency of v_1 assign the colors $2, m+4, \cdots, m+n$ to the vertices $u_3, \cdots, u_n$ of P_n. Hence $m+n$ colors is the minimum requirement. Also when $r = m+2$, $m+n$ colors are sufficient for proper r-coloring. The coloring in this case is same as the one given in Case 4 of first part. Thus $\chi_r(P_n\dot{\vee}P_m) = m+n$. □

Observation 1 For $n \geq 4$,

$$\chi_r(P_n\dot{\vee}P_2) = \begin{cases} r+2 & : \quad 1 \leq r \leq 3 \\ r+1 & : \quad 4 \leq r \leq \Delta \end{cases}$$

The minimum degree is $\delta(P_n\dot{\vee}P_2) = 2$ and maximum degree is $\Delta(P_n\dot{\vee}P_2) = n+1$.
Case 1 : When $1 \leq r \leq 3$.
The coloring for $r = 1$ is same as the one for $r = 1$ of Theorem 1.
When $r = 2$

$$f(u_i) = \begin{cases} 1, & when\ i\ is\ odd \\ 3, & when\ i\ is\ even \end{cases}$$

$f(a_i) = 2$ for $1 \leq i \leq n-1$
$f(v_1, v_2) = \{2, 4\}$
When $r = 3$
$f(u_1, u_2, \cdots, u_n) = \{1, 3, 2, 1, 3, 2, \cdots\}$
For $\{a_i : 1 \leq i \leq n-1\}$ provide suitable color from the set of colors $\{1, 2, 3\}$ so that each a_i satisfies 2-adjacency condition.
$f(v_1, v_2) = \{4, 5\}$
Case 2 : When $4 \leq r \leq \Delta = n+1$.
$f(u_1, u_2, \cdots, u_n) = \{1, 3, 2, 6, \cdots, r+1, 1, 3, 2, 6, \cdots, r+1, \cdots\}$. The coloring for remaining vertices are same as in Case 2.

Observation 2 For $m \geq 2$,

$$\chi_r(P_2\dot{\vee}P_m) = \Big\{ \; r+2 \quad : \quad 1 \leq r \leq m+1$$

The minimum degree is $\delta(P_2\dot{\vee}P_m) = 2$ and maximum degree is $\Delta(P_2\dot{\vee}P_m) = m+1$. The coloring for $r = 1$ is same as given in theorem 1.
When $2 \leq r \leq m+1$ the coloring is as follows:
$f(u_1, u_2) = \{1, 3\}$
$f(a_1) = 2$
$f(v_1, v_2, \cdots, v_m) = \{4, 5, 6, \cdots, r+2, 4, 5, 6, \cdots, r+2, \cdots\}$

Observation 3 For $m \geq 3$,

$$\chi_r(P_3\dot{\vee}P_m) = \begin{cases} r+2 & : \quad 1 \leq r \leq m+1 \\ m+3 & : \quad r = m+2 \end{cases}$$

The minimum degree is $\delta(P_3\dot{\vee}P_m) = 2$ and maximum degree is $\Delta(P_3\dot{\vee}P_m) = m+2$.
Case 1 : When $1 \leq r \leq m+1$ the coloring is as follows:
The coloring for $r = 1$ is same as given in theorem 1.
When $r = 2, 3$.
$f(u_1, u_2, u_3) = \{1, 3, 1\}$
$f(a_1) = 2$
$f(v_1, v_2, \cdots, v_m) = \{2, 4, \cdots, r+2, 2, 4, \cdots, r+2, \cdots\}$
When $4 \leq r \leq m+1$.
$f(u_1, u_2, u_3) = \{1, 3, 2\}$
$f(a_1, a_2) = \{2, 1\}$
$f(v_1, v_2, \cdots, v_m) = \{4, \cdots, r+2, 4, \cdots, r+2, \cdots\}$
Case 2 : When $r = m+2$ the coloring is same as the one for $r = m+1$.

Theorem 2.2. *For positive integers $n \geq 3, m \geq 2$, the r- dynamic chromatic number of subdivision - vertex join of path P_n with complete graph K_m is*

$$\chi_r(P_n\dot{\vee}K_m) = \begin{cases} m+1 & : \quad r = 1 \\ m+2 & : \quad 2 \leq r \leq m \\ m+3 & : \quad m+1 \leq r \leq m+2 \\ r+1 & : \quad m+3 \leq r \leq \Delta, n \geq 4 \end{cases}$$

Proof. The vertex set of $P_n\dot{\vee}K_m$ is $V(P_n\dot{\vee}K_m) = \{u_1, u_2, \cdots, u_n\} \cup \{a_1, a_2, \cdots, a_{n-1}\} \cup \{v_1, v_2, \cdots, v_m\}$. The edge set of $P_n\dot{\vee}K_m$ is $E(P_n\dot{\vee}K_m) = \{u_i a_i : 1 \leq i \leq n-1\} \cup \{u_i a_{i-1} : 2 \leq i \leq n\} \cup \{v_j v_k : 1 \leq j, k \leq m-1 \text{ and } j \neq k\} \cup \{u_i v_j : 1 \leq i \leq n, 1 \leq j \leq m\}$. The minimum degree, $\delta(P_n\dot{\vee}K_m) = 2$ and maximum degreee, $\Delta(P_n\dot{\vee}K_m) = m+n-1$.
Case 1 : When $r = 1$.
The vertices $\{u_i, v_1, v_2, \cdots, v_m\}$ induces a clique of order $m+1$ for all i and hence we have the lower bound $\chi_r(P_n\dot{\vee}K_m) \geq m+1$. We provide the upper bound $\chi_r(P_n\dot{\vee}K_m) \leq m+1$ using the color mapping $f : V(P_n\dot{\vee}K_m) \to \{1, 2, \cdots, m+1\}$ defined as follows.
$f(u_i) = 1$ for all $1 \leq i \leq n$
$f(a_i) = 2$ for $1 \leq i \leq n-1$ $f(v_1, v_2, \cdots, v_m) = \{2, 3, \cdots, m+1\}$
Hence $\chi_r(P_n\dot{\vee}K_m) = m+1$.

Case 2 : When $2 \leq r \leq m$.
Considering the vertices $a_i : 1 \leq i \leq n-1$ for satisfying its 2-adjacency we need to provide two different colors to its neighbors u_i and u_{i+1}. Thus let the colors $1, 2, 3$ be assigned to u_i, a_i, u_{i+1} respectively then the colors 1 and 3 cannot be assigned to any v_j's. So in order to color K_m we use the colors $2, 4, 5, \cdots, m+2$. And this coloring satisfies the r-adjacency of vertices till $r = m$. Thus $\chi_r(P_n\dot{\vee}K_m) \geq m+2$. Let $f : V(P_n\dot{\vee}K_m) \to \{1, 2, \cdots, m+2\}$ be the function

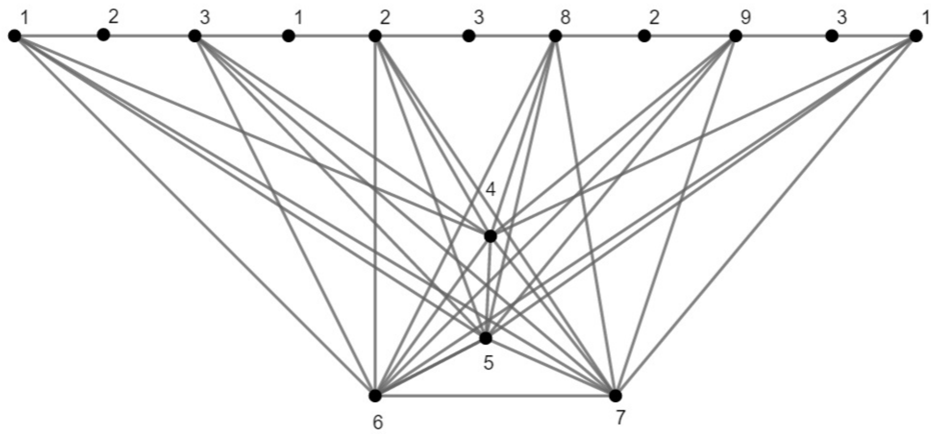

Figure 3. The 8-dynamic coloring of the graph $P_6\dot{\vee}K_4$

which defines the coloring in this case as below.

$$f(u_i) = \begin{cases} 1, \ when\ i\ is\ odd \\ 3, \ when\ i\ is\ even \end{cases}$$

$f(a_i) = 2$ for $1 \leq i \leq n-1$ $f(v_1, v_2, \cdots, v_m) = \{2, 4, \cdots, m+2\}$
Thus the upper bound is $\chi_r(P_n\dot{\vee}K_m) \leq m+2$ and we conclude that $\chi_r(P_n\dot{\vee}K_m) = m+2$.

Case 3 : When $m+1 \leq r \leq m+2$.
When $r = m+1$ we have by the lemma $\chi_r(P_n\dot{\vee}K_m) \geq min\,\{r, \Delta(P_n\dot{\vee}K_m)\} + 1 = r+1 = m+2$ but inorder to satisfy the r-adjacency of the vertices u_i we need an extra color $m+3$ in this case. Hence $\chi_r(P_n\dot{\vee}K_m) \geq m+3$. Again when $r = m+2$ we have by the lemma $\chi_r(P_n\dot{\vee}K_m) \geq min\,\{r, \Delta(P_n\dot{\vee}K_m)\} + 1 = r+1 = m+3$. We provide the coloring in this case by the mapping $f : V(P_n\dot{\vee}K_m) \to \{1, 2, \cdots, m+3\}$.

$$f(u_i) = \begin{cases} 1, \ when\ i \equiv 1(mod\ 3) \\ 3, \ when\ i \equiv 2(mod\ 3) \\ 2, \ when\ i \equiv 0(mod\ 3) \end{cases}$$

For $\{a_i : 1 \leq i \leq n-1\}$ provide suitable color from the set of colors $\{1, 2, 3\}$ so that each a_i satisfies 2-adjacency condition.
$f(v_1, v_2, \cdots, v_m) = \{4, 5, \cdots, m+3\}$
Thus $\chi_r(P_n\dot{\vee}K_m) = m+3$.

Case 4 : When $m+3 \leq r \leq \Delta, n \geq 4$.
For this case $n \geq 4$ because when $n = 3$ the maximum degree was $m+2$ and it ended in the previous case itself. By the lemma we the lower bound $\chi_r(P_n\dot{\vee}K_m) \geq min\,\{r, \Delta(P_n\dot{\vee}K_m)\} + 1 = r+1$. The upper bound is attained by the following coloring defined by the map $f : V(P_n\dot{\vee}K_m) \to \{1, 2, \cdots, r+1\}$.
$f(u_1, u_2, \cdots, u_n) = \{1, 3, 2, m+4, \cdots, r+1, 1, 3, 2, m+4, \cdots, r+1, \cdots\}$
For $a_i : 1 \leq i \leq n-1$ provide suitable color from the set of colors $\{1, 2, 3\}$ so that each a_i satisfies 2-adjacency condition.
$f(v_1, v_2, \cdots, v_m) = \{4, 5, \cdots, m+3\}$
Hence we have the upper bound $\chi_r(P_n\dot{\vee}K_m) \leq r+1$ and we conclude that $\chi_r(P_n\dot{\vee}K_m) = r+1$.
□

Observation 4 For $m \geq 2$,

$$\chi_r(P_2\dot{\vee}K_m) = \begin{cases} m+1 & : \quad r = 1 \\ m+2 & : \quad 2 \leq r \leq m \\ m+3 & : \quad r = m+1 \end{cases}$$

The minimum degree is $\delta(P_2\dot{\vee}K_m) = 2$ and maximum degree is $\Delta(P_2\dot{\vee}K_m) = m+1$. The coloring for cases 1, 2 and 3 are same as given in Case 1, 2 and 3 of theorem 2.

Theorem 2.3. *For positive integers n, m, the r- dynamic chromatic number of subdivision - vertex join of path P_n with star graph $K_{1,m}$ is*
***I.** When $n \geq 4, m \geq 2$ and $m+1 < n$*

$$\chi_r(P_n\dot{\vee}K_{1,m}) = \begin{cases} r+2 & : \quad 1 \leq r \leq 3 \\ 2r-1 & : \quad 4 \leq r \leq m+1, m \geq 3 \\ r+m & : \quad m+2 \leq r \leq n+1, m \geq 2 \\ m+n+1 & : \quad n+2 \leq r \leq m+n \end{cases}$$

Proof. Let the edge set of P_n be $E(P_n) = \{a_1, a_2, \cdots, a_{n-1}\}$. Then vertex set of $P_n\dot{\vee}K_{1,m}$ is $V(P_n\dot{\vee}K_{1,m}) = \{u_1, u_2, \cdots, u_n\} \cup \{a_1, a_2, \cdots, a_{n-1}\} \cup \{v_1, v_2, \cdots, v_{m+1}\}$ where v_1 is the central vertex of $K_{1,m}$ to which the m vertices are adjacent with. The edge set of $P_n\dot{\vee}K_{1,m}$ is $E(P_n\dot{\vee}K_{1,m}) = \{u_ia_i : 1 \leq i \leq n-1\} \cup \{u_ia_{i-1} : 2 \leq i \leq n\} \cup v_1v_j : 2 \leq j \leq m+1 \cup \{u_iv_j : 1 \leq i \leq n, 1 \leq j \leq m+1\}$. The minimum degree and maximum degree in this case is 2 and $m+n$ respectively.
Case 1 : When $1 \leq r \leq 3$.
Subcase 1 : $r = 1$.
The presence of cycle C_3 in $P_n\dot{\vee}K_{1,m}$ paves way to the fact that we require at least 3 different colors. Hence the lower bound $\chi_r(P_n\dot{\vee}K_{1,m}) \geq 3$. We provide the upper bound using the mapping $f : V(P_n\dot{\vee}K_{1,m}) \to \{1, 2, 3\}$ as follows:
$f(u_i) = 1$ for all i
$f(a_i) = 1$ for $1 \leq i \leq n-1$
$f(v_1, v_2, \cdots, v_{m+1}) = \{3, 2, 2, \cdots\}$
This gives the upper bound $\chi_r(P_n\dot{\vee}K_{1,m}) \leq 3$ and hence $\chi_r(P_n\dot{\vee}K_{1,m}) = 3 = r+2$.
Subcase 2 : $2 \leq r \leq 3$.
Consider the vertex a_1 which is of degree 2 in order to satisfy its 2-adjacency provide the colors 1, 2 and 3 to u_1, a_1, u_2 respectively. Now while considering the vertex u_1 for satisfying its 2-adjacency we need to provide a new color $4 = r+2$ to any one v_j since neither the color 1 and 3 can be applied to v_j's. Similarly when $r = 3$ for satisfying the r-adjacency condition of u_1 we need to provide the colors 4 and $5 = r+2$ to any of the two v_j's. Hence we require a minimum of $r+2$ different colors here i.e., $\chi_r(P_n\dot{\vee}K_{1,m}) \geq r+2$. The upper bound is given by the map $f : V(P_n\dot{\vee}K_{1,m}) \to \{1, 2, \cdots, r+2\}$.

$$f(u_i) = \begin{cases} 1, & when\ i\ is\ odd \\ 3, & when\ i\ is\ even \end{cases}$$

$f(a_i) = 2$ for $1 \leq i \leq n-1$
when $r = 2$, $f(v_1, v_2, \cdots, v_m + 1) = \{4, 2, 2, \cdots\}$
when $r = 3$, $f(v_1, v_2, \cdots, v_m + 1) = \{5, 4, 2, 2, \cdots\}$
Hence $\chi_r(P_n\dot{\vee}K_{1,m}) = r+2$.

Case 2 : When $4 \leq r \leq m+1, m \geq 3$.
Here in this case we consider $m \geq 3$ and the case when $m = 2$ does not come under this case because in this case the value of r varies from 4 to $m+1$ so it belongs to the next case. Consider the vertex u_1 and let be assigned the color 1 also let the vertices a_1, u_2 be assigned the colors 2 and 3 respectively. Now in order to satisfy the r-adjacency condition of u_1 we provide the colors $r+2, 4, \cdots, r+1$ to the vertices v_j's in order and this case ends at $r = m+1$. Now consider

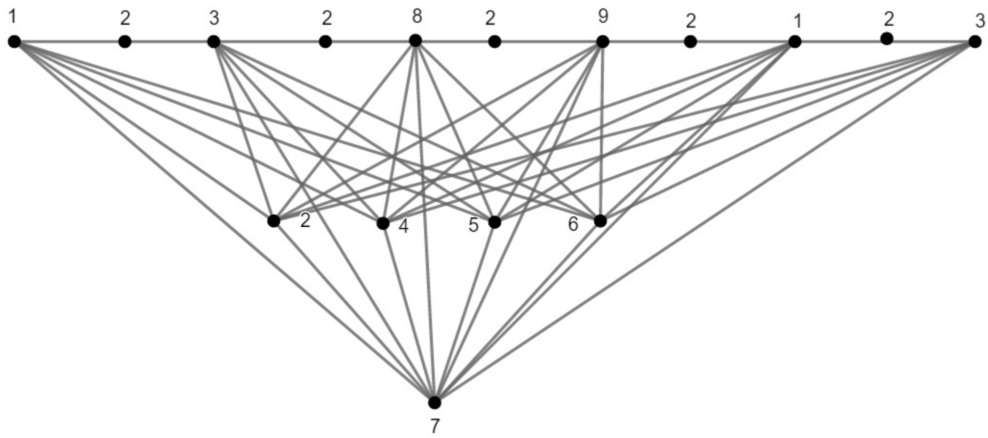

Figure 4. The 5-dynamic coloring of the graph $P_6\dot{\vee}K_{1,4}$

the vertex v_1 it is already adjacent to the vertex u_1, u_2 with colors 1 and 3 in order to satisfy the r-adjacency condition we need to provide the colors $r+3, \cdots, 2r-1$ to the remaining vertices of u_i since $m+1<n$. Hence we have the lower bound $\chi_r(P_n\dot{\vee}K_{1,m}) \geq 2r-1$. Consider the map $f: V(P_n\dot{\vee}K_{1,m}) \rightarrow \{1,2,\cdots,2r-1\}$ and the coloring is as below.
$f(u_1,u_2,\cdots,u_n)=\{1,3,r+3,\cdots,2r-1,1,3,r+3,\cdots,2r-1,\cdots\}$
$f(a_i)=2$ for $1\leq i\leq n-1$
$f(v_1,v_2,\cdots,v_{m+1})=\{r+2,2,4,\cdots,r+1,2,4,\cdots,r+1,\cdots\}$
This gives us the upper bound as $\chi_r(P_n\dot{\vee}K_{1,m}) \leq 2r-1$ and hence $\chi_r(P_n\dot{\vee}K_{1,m})=2r-1$.

Case 3 : When $m+2\leq r\leq n+1, m\geq 2$.
Let us first assign the vertices u_1,a_1,u_2 with the colors 1, 2, 3 respectively. Now the vertex u_1 with degree $m+2$ needs $m+2$ different colored neighbors hence assign the colors $4,\cdots,m+4$ colors to $v_2,v_3,\cdots,v_{m+1},v_1$. Now for satisfying the r-adjacency of the vertices v_j we provide the colors $m+5,\cdots,r+m$ to the remaining u_i's, $i\geq 3$. Thus we require a minimum of at least $r+m$ colors in this case hence $\chi_r(P_n\dot{\vee}K_{1,m})\geq r+m$. The coloring is given below using the map $f: V(P_n\dot{\vee}K_{1,m}) \rightarrow \{1,2,\cdots,r+m\}$.
When $m=2$ and $r=4$ the coloring is:
$f(u_1,u_2,\cdots,u_n)=\{1,3,2,1,3,2,\cdots\}$
$f(v_1,v_2,v_3)=\{4,5,6\}$ and for $\{a_i: 1\leq i\leq n-1\}$ provide suitable color from the set of colors $\{1,2,3\}$ so that each a_i satisfies 2-adjacency condition.
For all the remaining case the coloring is as below.
$f(u_1,u_2,\cdots,u_n)=\{1,3,m+5,\cdots,r+m,1,3,m+5,\cdots,r+m,\cdots\}$
For $\{a_i: 1\leq i\leq n-1\}$ provide the coloring as said for $m=2$ and $r=4$.
$f(v_1,v_2,\cdots,v_{m+1})=\{m+4,4,5,\cdots,m+3\}$
Thus we have the upper bound $\chi_r(P_n\dot{\vee}K_{1,m})\leq r+m$ and we conclude that $\chi_r(P_n\dot{\vee}K_{1,m})= r+m$.

Case 4 : When $n+2\leq r\leq m+n$.
By the case $r=n+1$ the r-adjacencies of all the vertices will be satisfied and we no longer require any new colors other than the $m+n$ colors used in the case $r=n+1$. The coloring in this case is as below.
$f(u_1,u_2,\cdots,u_n)=\{1,3,m+5,\cdots,m+n+1\}$
For the $\{a_i: 1\leq i\leq n-1\}$ provide suitable color from the set of colors $\{1,2,3\}$ so that each a_i satisfies 2-adjacency condition.
$f(v_1,v_2,\cdots,v_{m+1})=\{m+4,4,5,\cdots,m+3\}$
Hence $\chi_r(P_n\dot{\vee}K_{1,m})=m+n+1$. □

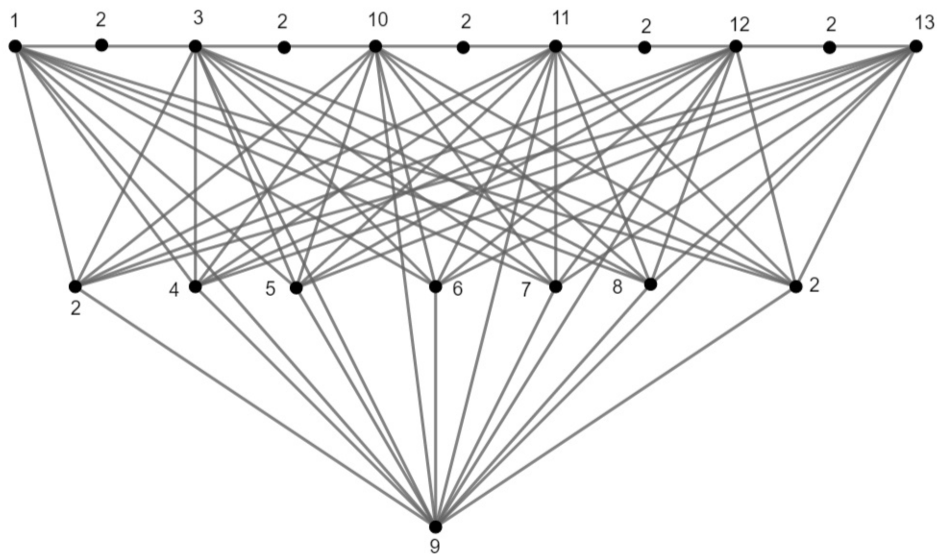

Figure 5. The 7-dynamic coloring of the graph $P_6\dot{\vee}K_{1,7}$

II. When $n \geq 4$ and $m+1 \geq n$

$$\chi_r(P_n\dot{\vee}K_{1,m}) = \begin{cases} r+2 & : \ 1 \leq r \leq 3 \\ 2r-1 & : \ 4 \leq r \leq n+1, m+1 > n \ and \ 4 \leq r \leq n, m+1 = n \\ r+n & : \ n+2 \leq r \leq m+1, m+1 \geq n+2 \\ m+n+1 & : \ m+2 \leq r \leq \Delta, m+1 \geq n \end{cases}$$

Proof. The maximum and minimum degrees in this case are $m+n$ and 2 respectively. The cases when $1 \leq r \leq 3$ is same as the one given in the earlier part of the theorem.
Case 2 : When $4 \leq r \leq n+1, m+1 > n \ and \ 4 \leq r \leq n, m+1 = n$.
When $m+1 = n$ the case ends at $r = n$ and the coloring for $r = n+1 = m+2$ goes to Case 4. Let us first assign the vertices u_1, a_1, u_2 with the colors 1, 2, 3 respectively. Now in order to satisfy the r-adjacency condition of u_1 we provide the colors $r+2, 4, \cdots, r+1$ to the vertices v_j's in order. Also while considering the vertex v_1 it is already adjacent to the vertex u_1, u_2 with colors 1 and 3 in order to satisfy the r-adjacency condition we need to provide the colors $r+3, \cdots, 2r-1$ to the remaining vertices of u_i and this case ends at $r = n+1$ since the degree of v_j is $n+1$ for $j \geq 2$. The coloring in this case is same as the one given in Case 2 of first part of the theorem. Thus $\chi_r(P_n\dot{\vee}K_{1,m}) = 2r-1$.

Case 3 : When $n+2 \leq r \leq m+1, m+1 \geq n+2$.
The condition $m+1 \geq n+2$ is necessary otherwise it will not be well-defined as in the case of $n = m+1$ and $m = n$. For $n = m+1$, $r = n+2 = m+3$ and $m = n$, $r = n+2 = m+2$ is provided in Case 4. Now for the remaining $m+1 \geq n+2$, in order to satisfy the r-adjacency condition of u_1 and v_i we provide the colors $r+2, 4, \cdots, r+1$ to the vertices v_j's in order and $1, 3, r+3, \cdots, r+n$ to the vertices u_i in order. Hence we have the lower bound $\chi_r(P_n\dot{\vee}P_m) \geq r+n$. The coloring is this case is defined by the mapping $f : V(P_n\dot{\vee}K_{1,m}) \rightarrow \{1, 2, \cdots, r+n\}$.
$f(u_1, u_2, \cdots, u_n) = \{1, 3, r+3, \cdots, r+n, 1, 3, r+3, \cdots, r+n, \cdots\}$
$f(a_i) = 2$
$f(v_1, v_2, \cdots, v_{m+1}) = \{r+2, 2, 4, 5, \cdots, r+1, 2, 4, 5, \cdots, r+1, \cdots\}$
Thus $\chi_r(P_n\dot{\vee}K_{1,m}) = r+n$.

Case 4 : When $m+2 \leq r \leq \Delta, m+1 \geq n$.
When $r = m+2$ for satisfying the $m+2$-adjacency of the vertex u_1 we first provide the colors

$m+4, 4, \cdots, m+3$ to the $m+1$ vertices of $K_{1,m}$ with the assumption that u_1, a_1, u_2 are assigned the color $1, 2, 3$. Now for satisfying the r-adjacency of v_1 assign the colors $2, m+5, \cdots, m+n+1$ to the vertices $u_3, \cdots, u_n$ of P_n. Hence $m+n+1$ colors is the minimum requirement. Also for all the remaining cases of r these $m+n+1$ colors are sufficient for proper r-coloring. The coloring in this case is same as the one given in Case 4 of first part. Thus $\chi_r(P_n \dot{\vee} K_{1,m}) = m+n+1$. □

Observation 5 For $m \geq 2$,

$$\chi_r(P_2 \dot{\vee} K_{1,m}) = \left\{ \begin{array}{lcl} r+2 & : & 1 \leq r \leq m+2 \end{array} \right.$$

The minimum degree is $\delta(P_2 \dot{\vee} K_{1,m}) = 2$ and maximum degree is $\Delta(P_2 \dot{\vee} K_{1,m}) = m+2$. The coloring for $r = 1$ is same as given for $r = 1$ of theorem 3.
When $2 \leq r \leq m+2$ the coloring is as follows:
$f(u_1, u_2) = \{1, 3\}$
$f(a_1) = 2$
when $2 \leq r \leq m+1$, $f(v_1, v_2, \cdots, v_{m+1}) = \{r+2, 2, 4, 5, 6, \cdots, r+1, 2, 4, 5, 6, \cdots, r+1, \cdots\}$
when $r = m+2$, $f(v_1, v_2, \cdots, v_{m+1}) = \{r+2, 4, 5, 6, \cdots, r+1, 4, 5, 6, \cdots, r+1, \cdots\}$

Observation 6 For $m \geq 2$,

$$\chi_r(P_3 \dot{\vee} K_{1,m}) = \left\{ \begin{array}{lcl} r+2 & : & 1 \leq r \leq m+2 \\ m+4 & : & r = m+3 \end{array} \right.$$

The minimum degree is $\delta(P_3 \dot{\vee} K_{1,m}) = 2$ and maximum degree is $\Delta(P_3 \dot{\vee} K_{1,m}) = m+3$.
Case 1 : When $1 \leq r \leq m+2$.
The coloring for $r = 1$ is same as given in theorem 3.
When $2 \leq r \leq m+2$ the coloring is as follows:
When $r = 2, 3$.
$f(u_1, u_2, u_3) = \{1, 3, 1\}$
$f(a_1) = 2$
$f(v_1, v_2, \cdots, v_{m+1}) = \{r+2, 2, 4, 5, 6, \cdots, r+1, 2, 4, 5, 6, \cdots, r+1, \cdots\}$
When $4 \leq r \leq m+1$.
$f(u_1, u_2, u_3) = \{1, 3, 2\}$
$f(a_1, a_2) = \{2, 1\}$
$f(v_1, v_2, \cdots, v_{m+1}) = \{r+2, 2, 4, 5, 6, \cdots, r+1, 2, 4, 5, 6, \cdots, r+1, \cdots\}$
Case 2 : When $r = m+3$.
$f(u_1, u_2, u_3) = \{1, 3, 2\}$
$f(a_1, a_2) = \{2, 1\}$
$f(v_1, v_2, \cdots, v_{m+1}) = \{m+4, 4, 5, 6, \cdots, m+3\}$.

References

[1] A.Ahadi, S.Akbari, A.Dehghana and M.Ghanbari, On the difference between chromatic and dynamic chromatic number of graphs, *Discrete Mathematics*, **312**(2012), 2579-2583.

[2] S.Akbari, M.Ghanbari and S.Jahanbakam, On the dynamic chromatic number of graphs,in:Combinatorics and Graphs, in:*Contemporary Mathematics,American Mathematical Society*, **531**(2010), 11-18.

[3] S.Akbari, M.Ghanbari and S.Jahanbakam, On the list dynamic coloring of graphs, *Discrete Applied Mathematics*, **157**(2009), 3005-3007.

[4] Ika Hesti Augestina, Dafik and A. Y. Harsyaa, On r-dynamic coloring of some graph operations, *Indonesian Journal of Combinatorics*, **1** (1) (2016), 22-30.

[5] G. Indulal, Spectrum of two new join of graphs and infinite families of integral graphs, *Kragujevac Journal of Mathematics*, **36**(2012), 133-139.

[6] H.J.Lai, B.Montgomery and H.Poon, Upper bounds of dynamic chromatic number, *Ars Combinatoria*, **68**(2003), 193-201.

[7] N. Mohanapriya, V.J.Vernold and M.Venkatachalam, On dynamic coloring of Fan graphs, *International Journal of Pure and Applied Mathematics*, **106**(8)(2016), 169-174.

[8] N. Mohanapriya, Ph.D thesis, A study on dynamic coloring of graphs, Bharathiar University, (2017), Coimbatore, India.

[9] B. Montgomery, Dynamic coloring of graphs, *ProQuest LLC, Ann Arbor*, MI, (2001), Ph.D Thesis, West Virginia University.

[10] A.Taherkhani, r-dynamic chromatic number of graphs, *Discrete Applied Mathematics*, **201**(2016), 222-227.

Author information

Aparna V, PG and Research Department of Mathematics, Kongunadu Arts and Science College, Coimbatore-641 029, Tamil Nadu, India.
E-mail: aparnav18794@gmail.com

Mohanapriya N, PG and Research Department of Mathematics, Kongunadu Arts and Science College, Coimbatore-641 029, Tamil Nadu, India.
E-mail: phdmohana@gmail.com

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