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A Study On $^{**}g\alpha$ -compactness and $^{**}g\alpha$-connectedness in Topological Spaces

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Abstract In this paper we introduce new concept of $^{**}g\alpha$ -compactness and $^{**}g\alpha$ -connectedness in Topological space using $^{**}g\alpha$-open sets and study some of their properties of $^{**}g\alpha$ -compactness and $^{**}g\alpha$ -connectedness.

1 Introduction

In 1991, Balachandran, Sundharam and Maki[1] introduced a class of compact space called GO-compact space and GO-connected space using g-open cover. In 2006, A.M. Shibani[12], introduce and studied about the rg-compact spaces and rg-connected spaces. In 2011, S.S. Benchalli and Priyanka M. Bansali[2] introduced the concept of gb-compactness and gb-connectedness Topological spaces and studied their basic properties. In 2016, S. Pious Missier and M. Anto[10] introduced the concept of generalize compactness and connectedness using g^*s- closed sets to obtained a weaker form of compactness and connectedness and studied the basic properties. In this paper, we introduce the $^{**}g\alpha$ -compactness, $^{**}g\alpha$ -connectedness in topological spaces and obtain some of its basic properties.

2 PRELIMINARIES

Let us recall the following definitions, which are useful in the sequel.

Definition 2.1. A subset A of a topological space (X, τ) is called
(1) a generalized closed set (briefly g-closed) [9] if cl⊆(A) U whenever A ⊆ U and U is open in (X, τ).
(2) a generalized α-closed set (briefly $g\alpha$-closed) [6] if αcl(A) ⊆ U whenever A ⊆ U and U is α-open in (X, τ).
(3) a gpr-closed[7] set if pcl(A)⊆ U whenever A ⊆ U and U is open in (X, τ).
(4) a $^*g\alpha$-closed set [14] if cl(A) ⊆ U whenever A ⊆ U and U is $g\alpha$-open in (X, τ).
(5) a $^{**}g\alpha$-closed set [15] if cl(A)⊆ U whenever A ⊆ U and U is $^*g\alpha$-open in (X, τ).

Definition 2.2. A function $f : (X, \tau) \longrightarrow (Y, \sigma)$ is called
(1) a g-continuous[1] if $f^{-1}(V)$ is an g-closed set of (X, τ) for every closed set V of (Y, σ).
(2) a gpr-continuous [8] if $f^{-1}(V)$ is a gpr-closed set of (X, τ) for every closed set V of (Y, σ).
(3) a $^*g\alpha$-continuous [14] if $f^{-1}(V)$ is a $^*g\alpha$-closed set of (X, τ) for every closed set V of (Y, σ).
(4) a $^{**}g\alpha$-continuous [13] if $f^{-1}(V)$ is a $^{**}g\alpha$-closed set of (X, τ) for every closed set V of (Y, σ).
(5) a $^{**}g\alpha$-irresolute [13] if $f^{-1}(V)$ is a $^{**}g\alpha$-closed set of (X, τ) for every $^{**}g\alpha$-closed set V of (Y, σ).

3 $^{**}g\alpha$ -COMPACTNESS

$^{**}g\alpha$- compactness is defined in this section and some of its characterizations are proved.

Definition 3.1. A collection $\{A_i : i \in \Delta\}$ of $^{**}g\alpha$-open sets in a topological space X is called a $^{**}g\alpha$-open cover of a subset S of X if $S \subset \bigcup\{A_i : i \in \Delta\}$ holds.

Definition 3.2. A topological space X is $^{**}g\alpha$ -compact, if every $^{**}g\alpha$-open cover of X has a finite sub cover.

Definition 3.3. A subset S of a topological space X is said to be $^{**}g\alpha$-compact relative to X, if for every collection $\{A_i : i \in \Delta\}$ of $^{**}g\alpha$ -open subsets of X such that $S \subset \bigcup\{A_i : i \in \Delta\}$ there exists a finite subset Δ_o of Δ such that $S \subset \bigcup\{A_i : i \in \Delta_o\}$.

Definition 3.4. A subset S of a topological space X is said to be $^{**}g\alpha$ -compact, if S is $^{**}g\alpha$-compact as a subspace of X.

Theorem 3.5. *Every $^{**}g\alpha$ -closed subset of a $^{**}g\alpha$ -compact space is $^{**}g\alpha$ -compact relative to X.*

Proof. Let A be a $^{**}g\alpha$-closed subset of a $^{**}g\alpha$ -compact space X. Then X - A is a $^{**}g\alpha$ -open in X. Let $M = \{G_\alpha : \alpha \in \Delta\}$ be a cover of A by $^{**}g\alpha$ -open sets in X. Then $M^* = M \cup A^c$ is a $^{**}g\alpha$ -open cover of X, i.e., $X = (\bigcup\{G_\alpha : \alpha \in \Delta\}) \cup A^c$. By hypothesis, X is $^{**}g\alpha$ -compact, hence M^* is reducible to a finite cover of X, say X = Gα $_1$∪ Gα_2∪ Gα_3∪....∪ G$_m$∪ A^c , Gα_k∈ M. But A and A^c are disjoint, Hence A ⊂ Gα $_1$∪ Gα_2∪ Gα_3∪....∪ G$_m$∪ A^c , Gα_k∈ M. But we have shown that any $^{**}g\alpha$ -open cover M of A contains a finite sub cover, i.e., A is $^{**}g\alpha$-compact relative to X.

Theorem 3.6. *A $^{**}g\alpha$ -continuous image of a $^{**}g\alpha$ -compact space is compact.*

Proof. Let $f : X \longrightarrow Y$ be a $^{**}g\alpha$ -continuous map from a $^{**}g\alpha$ -compact space X on a topological space Y. Let $\{A_i : i \in \Delta\}$ be an open cover of Y. Then $\{f^{-1}(A_i) : i \in \Delta\}$is a $^{**}g\alpha$ -open cover of X. Since X is $^{**}g\alpha$ -compact, it has a finite sub cover, say $\{f^{-1}(A_1), ..., f^{-1}(A_n)\}$. Since f is onto $\{A_1, A_2, ..., A_n\}$ is a cover of Y which is finite. Therefore Y is compact.

Theorem 3.7. *If a map $f : X \longrightarrow Y$ is $^{**}g\alpha$ -irresolute and a subset B of X is $^{**}g\alpha$ -compact relative to X, then the image f(B) is $^{**}g\alpha$ -compact relative to Y.*

Proof. Let $\{A_i : i \in \Delta_0\}$ be any collection of $^{**}g\alpha$ -open subsets of Y such that $f(B) \subset \bigcup\{A_i : i \in \Delta\}$. Then $B \subset \bigcup\{f^{-1}(A_i) : i \in \Delta\}$ holds. By hypothesis there exists a finite subset Δ_o such that $B \subset \bigcup\{f^{-1}(A_i) : i \in \Delta_o\}$. Therefore we have $f(B) \subset \bigcup\{(A_i : i \in \Delta_o\}$, which shows that f (B) is a $^{**}g\alpha$ -compact relative to Y.

Theorem 3.8. *The product space of two non empty spaces is $^{**}g\alpha$ -compact, then each factor space is $^{**}g\alpha$ -compact.*

Proof. Let $X \times Y$ be the product space of the non empty spaces X and Y and suppose $X \times Y$ is a $^{**}g\alpha$ -compact. Then the projection $\prod : X \times Y \longrightarrow X$ is a $^{**}g\alpha$ -irresolute map. Hence $\prod(X \times Y) = X$ is $^{**}g\alpha$ -compact. Similarly we prove for the space Y.

Theorem 3.9. *Every $^{**}g\alpha$ -compact space is compact.*

Proof. Let (X, τ) be a $^{**}g\alpha$ -compact space. Let $\{B_\alpha : \alpha \in \Delta\}$ be an open cover of X. Then $X = \{B_\alpha : \alpha \in \Delta\}$. Since every open set is $^{**}g\alpha$ -open, so $\{B_\alpha : \alpha \in \Delta\}$ is a $^{**}g\alpha$-open cover of X. Since X is $^{**}g\alpha$-compact, it has a finite subcover, say $\{B_1, B_2, B_3, ..., B_n\}$. Hence, X is compact.

Theorem 3.10. *A space X is $^{**}g\alpha$ -compact if and only if each family of $^{**}g\alpha$-closed subsets of X with the finite intersection property has a non-empty intersection .*

Proof. Given collection A of subsets of X, let $S = \{X - A : A \in \Delta\}$ be the collection of their complements. Then the following statements hold.
(i) A is a collection of $^{**}g\alpha$-open sets if and only if S is a collection of $^{**}g\alpha$-closed sets.
(ii) The collection A covers X if and only if the intersection $\bigcap_{c \in S}$S of all the elements of S is empty.

(iii) The finite sub collection $\{A_1, A_2, ...A_n\}$ of A covers X if and only if the intersection of the corresponding elements $S_i = X - A_i$ of S is empty. the statement (i) is trivial, while the (ii) and (iii) follow from De Morgans law. X - $(\bigcup_{\alpha \in J})$ = $\bigcap_{\alpha \in J}(X - A_\alpha)$.The proof of the theorem now proceeds in two steps, taking contra positive of the theorem and then the complement. the statement X is $^{**}g\alpha$-compact is equivalent to: Given any collection A of $^{**}g\alpha$-open subsets of X, if A covers X, then some finite sub collection of A covers X. This statement is equivalent to its contra positive, which is the following.
Given any collection S of $^{**}g\alpha$-closed sets, if every finite intersection of elements of S is not-empty, then the intersection of all the elements of S is non-empty. This is the just condition of our theorem.

4 $^{**}g\alpha$ -CONNECTEDNESS

Definition 4.1. A topological space X is said to be $^{**}g\alpha$ -connected, if X canot be written as a disjoint union of two non empty $^{**}g\alpha$-open sets. A subset of X is $^{**}g\alpha$ -connected if it is $^{**}g\alpha$ -connected as a subspace.

Theorem 4.2. *For a topological space X the following are equivalent:*
(i) X is $^{**}g\alpha$ -connected.
(ii) X and ϕ are the only subsets of X which are both $^{**}g\alpha$ -open and $^{**}g\alpha$ -closed.
(iii) Each $^{**}g\alpha$ -continuous map of X into a discrete space Y with at least two points is a constant map.

Proof.
$(i) \rightarrow (ii)$: Let A be a $^{**}g\alpha$ -open and $^{**}g\alpha$ -closed subset of X. Then A^c is both $^{**}g\alpha$ -closed and $^{**}g\alpha$ -open. Since X is the disjoint union of the $^{**}g\alpha$ -open sets A and A^c, one of these must be empty. That is $A = \phi$ or A = X .
$(ii) \rightarrow (i)$: Suppose that $X = A \cup B$, where A and B are disjoint non-empty $^{**}g\alpha$ -open subsets of X. Then A is both $^{**}g\alpha$ -open and $^{**}g\alpha$ -closed. By assumption, $A = \phi$ or A = X. Therefore X is $^{**}g\alpha$ -connected.
$(ii) \rightarrow (iii)$: Let $f : X \longrightarrow Y$ is a $^{**}g\alpha$ -continuous map then X is covered by $^{**}g\alpha$ -open and $^{**}g\alpha$ -closed covering $\{f^{-1}(y) : y \in Y\}$. By assumption f^{-1} (y) = ϕ or X for each. If $f^{-1}(y) = \phi$ for all $y \in Y$, then f fails to be map. Then, there exists only one point $y \in Y$ such that $f^{-1}(y) \neq \phi$ and hence $f^{-1}(y) = X$. This show that f is a constant map.
$(iii) \rightarrow (ii)$: Let A by both $^{**}g\alpha$ -open and $^{**}g\alpha$ -closed set in X. Suppose $A \neq \phi$. Let $f : X \longrightarrow Y$ is a $^{**}g\alpha$ -continuous map defined by $f(A) = y$ and $f(A^c) = w$ for some distinct points y and w in Y. By assumption f is constant. Therefore we have A = X.

Theorem 4.3. *Every $^{**}g\alpha$ -connected space is connected but the converse need not be true.*

Proof. Let (X, τ) be a $^{**}g\alpha$- connected space. Suppose that (X, τ) is not connected. Then X = A $\cup$ B, where A and B are disjoint nonempty open sets in (X, τ). We know that arbitrary union of $^{**}g\alpha$ -open sets is $^{**}g\alpha$ -open, A and B are $^{**}g\alpha$ -open and X = A $\cup$ B, where A and B are disjoint nonempty and $^{**}g\alpha$ -open sets in (X, τ). This contradicts the fact that (X, τ) is $^{**}g\alpha$ -connected and so (X, τ) is connected.

Example 4.4. Let X = $\{a, b, c\}$, τ = $\{\phi, X, \{a, b\}, \{a\}\}$, then (X, τ) is not $^{**}g\alpha$ -connected, because every subsets of X is $^{**}g\alpha$ -open. The only clopen sets of X are ϕ and X. Therefore X is connected.

Theorem 4.5. *(i) If f: X $\longrightarrow$ Y is a $^{**}g\alpha$ -continuous surjection and X is $^{**}g\alpha$ -connected, then Y is connected.*
(ii) If f: X $\longrightarrow$ Y is a $^{**}g\alpha$ -irresolute surjection and X is $^{**}g\alpha$ -connected, then Y is $^{**}g\alpha$ -connected.

Proof. (i): Suppose that Y is not connected. Let Y = A $\cup$ B, where A and B are disjoint non-empty open sets in Y. Since f is $^{**}g\alpha$ -continuous and onto, $X = f^{-1}(A) \cup f^{-1}(B)$, where $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint non-empty and $^{**}g\alpha$ -open sets in X. This contradicts the fact that X is $^{**}g\alpha$ connected. Hence Y is connected.

(ii): Suppose that Y is not $^{**}g\alpha$ -connected. Let Y = A ∪ B, where A and B are disjoint non empty and $^{**}g\alpha$ -open sets in Y. Since f is $^{**}g\alpha$ -irresolute and onto, $X = f^{-1}(A)\cup$ f$^{-1}(B)$, where $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint non-empty and $^{**}g\alpha$ -open sets in X. This is a contradiction to the fact that X is $^{**}g\alpha$ -connected. Hence Y is $^{**}g\alpha$ -connected.

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