A Study On $^{∗∗}gα$-compactness and $^{∗∗}gα$-connectedness in Topological Spaces

A. Singaravelan

Communicated by V. Kokilavani

AMS 2010 Classifications: 54B05, 54B10, 54C10, 54D18.

Keywords and phrases: $^{∗∗}gα$-closed set, $^{∗∗}gα$-open set, $^{∗∗}gα$-irresolute, $^{∗∗}gα$-compactness, $^{∗∗}gα$-connectedness.

Abstract In this paper we introduce new concept of $^{∗∗}gα$-compactness and $^{∗∗}gα$-connectedness in Topological space using $^{∗∗}gα$-open sets and study some of their properties of $^{∗∗}gα$-compactness and $^{∗∗}gα$-connectedness.

1 Introduction

In 1991, Balachandran, Sundharam and Maki[1] introduced a class of compact space called GO-compact space and GO-connected space using g-open cover. In 2006, A.M. Shibani[12], introduce and studied about the rg-compact spaces and rg-connected spaces. In 2011, S.S. Benchalli and Priyanka M. Bansali[2] introduced the concept of gb-compactness and gb-connectedness Topological spaces and studied their basic properties. In 2016, S. Pious Missier and M. Anto[10] introduced the concept of generalize compactness and connectedness using $g^∗s$- closed sets to obtained a weaker form of compactness and connectedness and studied the basic properties. In this paper, we introduce the $^{∗∗}gα$-compactness, $^{∗∗}gα$-connectedness in topological spaces and obtain some of its basic properties.

2 PRELIMINARIES

Let us recall the following definitions, which are useful in the sequel.

Definition 2.1. A subset A of a topological space $(X, τ)$ is called
(1) a generalized closed set (briefly g-closed) [9] if $cl(A) U$ whenever $A U$ and $U$ is open in $(X, τ)$.
(2) a generalized $α$-closed set (briefly $gα$-closed) [6] if $αcl(A) U$ whenever $A U$ and $U$ is $α$-open in $(X, τ)$.
(3) a gpr-closed[7] set if $pcl(A) U$ whenever $A U$ and $U$ is open in $(X, τ)$.
(4) a $^{∗}gα$-closed set [14] if $cl(A) U$ whenever $A U$ and $U$ is $gα$-open in $(X, τ)$.
(5) a $^{∗∗}gα$-closed set [15] if $cl(A) U$ whenever $A U$ and $U$ is $^{∗}gα$-open in $(X, τ)$.

Definition 2.2. A function $f : (X, τ) → (Y, σ)$ is called
(1) a g-continuous[1] if $f^{-1}(V)$ is an g-closed set of $(X, τ)$ for every closed set $V$ of $(Y, σ)$.
(2) a gpr-continuous [8] if $f^{-1}(V)$ is a gpr-closed set of $(X, τ)$ for every closed set $V$ of $(Y, σ)$.
(3) a $^{∗}gα$-continuous [14] if $f^{-1}(V)$ is a $^{∗}gα$-closed set of $(X, τ)$ for every closed set $V$ of $(Y, σ)$.
(4) a $^{∗∗}gα$-continuous [13] if $f^{-1}(V)$ is a $^{∗∗}gα$-closed set of $(X, τ)$ for every closed set $V$ of $(Y, σ)$.
(5) a $^{∗∗}gα$-irresolute [13] if $f^{-1}(V)$ is a $^{∗∗}gα$-closed set of $(X, τ)$ for every $^{∗∗}gα$-closed set $V$ of $(Y, σ)$.

3 $^{∗∗}gα$-COMPACTNESS

$^{∗}gα$-compactness is defined in this section and some of its characterizations are proved.
Definition 3.1. A collection \( \{A_i : i \in \Delta \} \) of \(*^\ast\) -open sets in a topological space \( X \) is called a \(*^\ast\) -open cover of a subset \( S \) of \( X \) if \( S \subset \bigcup \{A_i : i \in \Delta \} \) holds.

Definition 3.2. A topological space \( X \) is \(*^\ast\) -compact, if every \(*^\ast\) -open cover of \( X \) has a finite sub cover.

Definition 3.3. A subset \( S \) of a topological space \( X \) is said to be \(*^\ast\) -compact relative to \( X \), if for every collection \( \{A_i : i \in \Delta \} \) of \(*^\ast\) -open subsets of \( X \) such that \( S \subset \bigcup \{A_i : i \in \Delta \} \) there exists a finite subset \( \Delta_0 \) of \( \Delta \) such that \( S \subset \bigcup \{A_i : i \in \Delta_0 \} \).

Definition 3.4. A subset \( S \) of a topological space \( X \) is said to be \(*^\ast\) -compact, if \( S \) is \(*^\ast\) -compact as a subspace of \( X \).

Theorem 3.5. Every \(*^\ast\) -closed subset of a \(*^\ast\) -compact space is \(*^\ast\) -compact relative to \( X \).

Proof. Let \( A \) be a \(*^\ast\) -closed subset of a \(*^\ast\) -compact space \( X \). Then \( X - A \) is a \(*^\ast\) -open in \( X \). Let \( M = \{G_\alpha : \alpha \in \Delta \} \) be a cover of \( A \) by \(*^\ast\) -open sets in \( X \). Then \( M^* = M \cup A^c \) is a \(*^\ast\) -open cover of \( X \), i.e., \( X = (\bigcup \{G_\alpha : \alpha \in \Delta \}) \cup A^c \). By hypothesis, \( X \) is \(*^\ast\) -compact, hence \( M^* \) is reducible to a finite cover of \( X \), say \( X = G_\alpha \cup G_\alpha \cup G_\alpha \cup ... \cup G_\alpha \cup A^c \), \( G_\alpha \in M \). But \( A \) and \( A^c \) are disjoint, Hence \( A \subset G_\alpha \cup G_\alpha \cup G_\alpha \cup ... \cup G_\alpha \cup A^c \), \( G_\alpha \in M \). But we have shown that any \(*^\ast\) -open cover \( M \) of \( A \) contains a finite sub cover, i.e., \( A \) is \(*^\ast\) -compact relative to \( X \).

Theorem 3.6. A \(*^\ast\) -continuous image of a \(*^\ast\) -compact space is compact.

Proof. Let \( f : X \rightarrow Y \) be a \(*^\ast\) -continuous map from a \(*^\ast\) -compact space \( X \) on a topological space \( Y \). Let \( \{A_i : i \in \Delta \} \) be an open cover of \( Y \). Then \( \{f^{-1}(A_i) : i \in \Delta \} \) is a \(*^\ast\) -open cover of \( X \). Since \( X \) is \(*^\ast\) -compact, it has a finite sub cover, say \( \{f^{-1}(A_1), \ldots, f^{-1}(A_n)\} \).

Since \( f \) is onto \( \{A_1, A_2, \ldots, A_n \} \) is a cover of \( Y \) which is finite. Therefore \( Y \) is compact.

Theorem 3.7. If a map \( f : X \rightarrow Y \) is \(*^\ast\) -irresolute and a subset \( B \) of \( X \) is \(*^\ast\) -compact relative to \( X \), then the image \( f(B) \) is \(*^\ast\) -compact relative to \( Y \).

Proof. Let \( \{A_i : i \in \Delta_0 \} \) be any collection of \(*^\ast\) -open subsets of \( Y \) such that \( f(B) \subset \bigcup \{A_i : i \in \Delta \} \) holds. By hypothesis there exists a finite subset \( \Delta_0 \), such that \( B \subset \bigcup \{f^{-1}(A_i) : i \in \Delta_0 \} \). Therefore we have \( f(B) \subset \bigcup \{f^{-1}(A_i) : i \in \Delta_0 \} \), which shows that \( f(B) \) is \(*^\ast\) -compact relative to \( Y \).

Theorem 3.8. The product space of two non-empty spaces is \(*^\ast\) -compact, then each factor space is \(*^\ast\) -compact.

Proof. Let \( X \times Y \) be the product space of the non-empty spaces \( X \) and \( Y \) and suppose \( X \times Y \) is a \(*^\ast\) -compact. Then the projection \( \prod : X \times Y \rightarrow X \) is a \(*^\ast\) -irresolute map. Hence \( \prod((X \times Y)) = X \) is \(*^\ast\) -compact. Similarly we prove for the space \( Y \).

Theorem 3.9. Every \(*^\ast\) -compact space is compact.

Proof. Let \( (X, \tau) \) be a \(*^\ast\) -compact space. Let \( \{B_\alpha : \alpha \in \Delta \} \) be an open cover of \( X \). Then \( X = \{B_\alpha : \alpha \in \Delta \} \). Since every open set is \(*^\ast\) -open, so \( \{B_\alpha : \alpha \in \Delta \} \) is a \(*^\ast\) -open cover of \( X \). Since \( X \) is \(*^\ast\) -compact, it has a finite subcover, say \( \{B_1, B_2, B_3, \ldots, B_n \} \). Hence, \( X \) is compact.

Theorem 3.10. A space \( X \) is \(*^\ast\) -compact if and only if each family of \(*^\ast\) -closed subsets of \( X \) with the finite intersection property has a non-empty intersection.

Proof. Given collection \( A \) of subsets of \( X \), let \( S = \{X - A : A \in \Delta \} \) be the collection of their complements. Then the following statements hold.

(i) \( A \) is a collection of \(*^\ast\) -open sets if and only if \( S \) is a collection of \(*^\ast\) -closed sets.

(ii) The collection \( A \) covers \( X \) if and only if the intersection \( \bigcap_{c \in S} S \) is empty.
(iii) The finite sub collection \( \{A_1, A_2, ..., A_n\} \) of \( A \) covers \( X \) if and only if the intersection of the corresponding elements \( S_i = X - A_i \) of \( S \) is empty. The statement (i) is trivial, while the (ii) and (iii) follow from De Morgans law. \( X - (\bigcup_{i \in J} (X - A_i)) \). The proof of the theorem now proceeds in two steps, taking contra positive of the theorem and then the complement. The statement \( X \) is \( **g_\alpha \)-compact is equivalent to: Given any collection \( A \) of \( **g_\alpha \)-open subsets of \( X \), if \( A \) covers \( X \), then some finite sub collection of \( A \) covers \( X \). This statement is equivalent to its contra positive, which is the following.

Given any collection \( S \) of \( **g_\alpha \)-closed sets, if every finite intersection of elements of \( S \) is not-empty, then the intersection of all the elements of \( S \) is not-empty. This is the just condition of our theorem.

4 **\( g_\alpha \) -CONNECTEDNESS

**Definition 4.1.** A topological space \( X \) is said to be \( **g_\alpha \) -connected, if \( X \) can not be written as a disjoint union of two non empty \( **g_\alpha \)-open sets. A subset of \( X \) is \( **g_\alpha \) -connected if it is \( **g_\alpha \) -connected as a subspace.

**Theorem 4.2.** For a topological space \( X \) the following are equivalent:

(i) \( X \) is \( **g_\alpha \) -connected.

(ii) \( X \) and \( \phi \) are the only subsets of \( X \) which are both \( **g_\alpha \) -open and \( **g_\alpha \) -closed.

(iii) Each \( **g_\alpha \) -continuous map of \( X \) into a discrete space \( Y \) with at least two points is a constant map.

**Proof.**

(i) \( \rightarrow \) (ii): Let \( A \) be a \( **g_\alpha \) -open and \( **g_\alpha \) -closed subset of \( X \). Then \( A^c \) is both \( **g_\alpha \) -closed and \( **g_\alpha \) -open. Since \( X \) is the disjoint union of the \( **g_\alpha \) -open sets \( A \) and \( A^c \), one of these must be empty. That is \( A = \phi \) or \( A = X \).

(ii) \( \rightarrow \) (i): Suppose that \( X = A \cup B \), where \( A \) and \( B \) are disjoint non-empty \( **g_\alpha \) -open subsets of \( X \). Then \( A \) is both \( **g_\alpha \) -open and \( **g_\alpha \) -closed. By assumption, \( A = \phi \) or \( A = X \). Therefore \( X \) is \( **g_\alpha \) -connected.

(iii) \( \rightarrow \) (ii): Let \( f : X \rightarrow Y \) be a \( **g_\alpha \) -continuous map then \( X \) is covered by \( **g_\alpha \) -open and \( **g_\alpha \) -closed covering \( \{f^{-1}(y) : y \in Y \} \). By assumption \( f^{-1}(y) = \phi \) or \( X \) for each. If \( f^{-1}(y) = \phi \) for all \( y \in Y \), then \( f \) fails to be map. Then, there exists only one point \( y \in Y \) such that \( f^{-1}(y) \neq \phi \) and hence \( f^{-1}(y) = X \). This show that \( f \) is a constant map.

(iii) \( \rightarrow \) (i): Let \( A \) be both \( **g_\alpha \) -open and \( **g_\alpha \) -closed subset of \( X \). Suppose \( A \neq \phi \). Let \( f : X \rightarrow Y \) be a \( \ast g_\alpha \) -continuous map defined by \( f(A) = y \) and \( f(A^c) = w \) for some distinct points \( y \) and \( w \) in \( Y \). By assumption \( f \) is constant. Therefore we have \( A = X \).

**Theorem 4.3.** Every \( **g_\alpha \) -connected space is connected but the converse need not be true.

**Proof.** Let \( (X, \tau) \) be a \( **g_\alpha \) -connected space. Suppose that \( (X, \tau) \) is not connected. Then \( X = A \cup B \), where \( A \) and \( B \) are disjoint nonempty open sets in \( (X, \tau) \). We know that arbitrary union of \( **g_\alpha \) -open sets is \( **g_\alpha \) -open, and \( A \) and \( B \) are \( **g_\alpha \) -open and \( X = A \cup B \), where \( A \) and \( B \) are disjoint nonempty and \( **g_\alpha \) -open sets in \( (X, \tau) \). This contradicts the fact that \( (X, \tau) \) is \( **g_\alpha \) -connected and so \( (X, \tau) \) is connected.

**Example 4.4.** Let \( X = \{a, b, c\}, \tau = \{\phi, X, \{a, b\}, \{a\}\} \), then \( (X, \tau) \) is not \( **g_\alpha \) -connected, because every subsets of \( X \) is \( **g_\alpha \) -open. The only clopen sets of \( X \) are \( \phi \) and \( X \). Therefore \( X \) is connected.

**Theorem 4.5.** (i) If \( f : X \rightarrow Y \) is a \( **g_\alpha \) -continuous surjection and \( X \) is \( **g_\alpha \) -connected, then \( Y \) is connected.

(ii) If \( f : X \rightarrow Y \) is a \( **g_\alpha \) -irresolute surjection and \( X \) is \( **g_\alpha \) -connected, then \( Y \) is \( **g_\alpha \) -connected.

**Proof.** (i): Suppose that \( Y \) is not connected. Let \( Y = A \cup B \), where \( A \) and \( B \) are disjoint non-empty open sets in \( Y \). Since \( f \) is \( **g_\alpha \) -continuous and onto, \( X = f^{-1}(A) \cup f^{-1}(B) \), where \( f^{-1}(A) \) and \( f^{-1}(B) \) are disjoint non-empty and \( **g_\alpha \) -open sets in \( X \). This contradicts the fact that \( X \) is \( **g_\alpha \) -connected. Hence \( Y \) is connected.
(ii): Suppose that $Y$ is not $^{**}g\alpha$-connected. Let $Y = A \cup B$, where $A$ and $B$ are disjoint non empty and $^{**}g\alpha$-open sets in $Y$. Since $f$ is $^{**}g\alpha$-irresolute and onto, $X = f^{-1}(A) \cup f^{-1}(B)$, where $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint non-empty and $^{**}g\alpha$-open sets in $X$. This is a contradiction to the fact that $X$ is $^{**}g\alpha$-connected. Hence $Y$ is $^{**}g\alpha$-connected.

References


Author information

A. Singaravelan, Department of Mathematics, Kongunadu Arts and Science College(Autonomous), Coimbatore-641029, Tamilnadu, India.
E-mail: singaravelna_ma@kongunaducollege.ac.in

Received : January 4 2021
Accepted : March 22, 2021