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The Vertex Irregular Reflexive Labeling of Some Almost Regular Graph

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Abstract A labeling of a graph is a mapping of graphs (vertices or edges) into a set of positive integers or a set of non-negative integers. Let H be a connected, simple, nontrivial, and un-directed graph with the vertex set $V(H)$ and the edge set $E(H)$. A total k-labeling is a function f_e from $E(H)$ to first natural number k_e and a function f_v from $V(H)$ to non negative even number up to $2k_v$, where $k = max\{k_e, 2k_v\}$. A *vertex irregular reflexive k-labeling* of the graph G is total k-labeling, if for every two different vertices have different weight, where the weight of a vertex is the sum of labels of edges which are incident this vertex and the vertex label itself. The reflexive vertex strength of the graph G, denoted by $rvs(G)$ is a minimum k such that graph G has a vertex irregular reflexive k-labelling. In this paper, We will determine the exact value of reflexive vertex strength on ladder graph and bipartite complete $(K_{2,n})$.

1 Introduction

A graph H is a pair set $(V(H), E(H))$ with $V(H)$ is a nonempty set and $E(H)$ (may empty) is a non ordered set of two different elements in $V(H)$. The element of $V(H)$ is called the vertex of H and the element of $E(H)$ is called the edge of H. The two vertices x and y in H are called connected if graph H contains a path $x - y$. The graph H is called a connected graph if every two different vertices in H are connected. The cardinality of vertex set $V(H)$ is called order and denoted by p. The cardinality of the edge set $E(H)$ is called size and denoted by q. The number of edges which are incident to a vertex is called degree. The maximum degree of graph H is denoted by $\Delta(H)$ and the minimum degree of graph H is denoted by $\delta(H)$ [1].

According to Gallian in 2017 [2], labeling is a mapping of graphs (vertices or edges) into a set of positive integers or a set of non-negative integers. Types of labeling are classified based on their domain. If the domain is a set of vertices or a set of edges, then the labeling is named as vertex labeling or edge labeling, respectively. If the domain is a combination of both vertex set and edge set, then it is called total labeling.

Bača et al. [13] introduced total k irregular labeling by the following definition. For any positive integer k, the total k irregular labeling is a function $g : V(H) \cup E(H) \longrightarrow \{1, 2, 3, 4, 5, \ldots, k\}$. The total k irregular labeling consists of two types, namely a vertex total k irregular labeling and an edge total k irregular labeling. In the total k irregular labeling, the vertex weight is the sum of the vertex labels and all incident edge labels at that vertex. The edge weight is the sum of the edge labels and the two corresponding vertex labels on that edge. The vertex total k irregular labeling is a mapping of domain vertex and edge set of graph H to the set of positive integers such that the weight of each vertex is distinct. The total vertex irregularity strength of graph H, denoted by $tvs(H)$ is minimum k such that graph H has a vertex total k irregular labeling. The lower bound of $tvs(H)$ and some results on total vertex irregularity strength of some graphs can be found in [3, 4, 5, 6, 7, 8, 9, 10].

Irregular labeling was developed on the irregular reflexive k labeling by Tanna et al. in 2018. Tanna et al. determined the reflexive edge strength on prism graphs (D_n), wheel graphs (W_n), fan graphs (F_n) and basket graphs (B_n) and determined the lower bound for reflexive edge strength of any graph H. From this research, they gave some open problems, namely determining the vertex irregular reflexive of some graphs. The formal definition of the vertex irregular reflexive was introduced by Tanna et al. as follows [12].

Let H be a connected graph under the function $f_e : E(H) \longrightarrow \{1, 2, 3, \ldots, k_e\}$ and $f_v : V(H) \longrightarrow \{0, 2, 4, \ldots, 2k_v\}$ where $k = max\{k_e, 2k_v\}$. The vertex irregular reflexive k-labeling of graph H satisfies $W(x) \neq W(x')$ for $\forall\, x, x' \in V(H)$, where $W(x) = f_v(x) + \Sigma_{xx' \in E(H)} f_e(xx')$. The reflexive vertex strength of graph H, denoted by $rvs(H)$ is minimum k such that graph H has a vertex irregular reflexive k-labeling. The following Lemma can be used to determined the lower bound of $rvs(H)$ [12].

Lemma 1.1. *[12] Let H be a graph of order p and minimum degree δ. The largest vertex weight of graph H under any vertex irregular reflexive k-labeling is at least:*

(i) $p + \delta - 1$, if $p \equiv 0$ (modulo 4), $p \equiv 1$ (modulo 4), and $\delta \equiv 0$ (modulo 2) or $p \equiv 3$ (modulo 4) and $\delta \equiv 1$ (modulo 2),

(ii) $p + \delta$, otherwise.

Based on the Lemma 1.1 above obtained the following Corollary.

Corollary 1.2. *[12] Let H' be a α-regular graph of order n. The reflexive vertex strength of graph H' is*

$$rvs(H') = \begin{cases} \left\lceil \dfrac{n+\alpha-1}{\alpha+1} \right\rceil, & \text{jika } n \equiv 0, 1 \pmod 4, \\ \\ \left\lceil \dfrac{n+\alpha}{\alpha+1} \right\rceil, & \text{jika } n \equiv 2, 3 \pmod 4. \end{cases}$$

Agustin et al. [11] have determined the exact value of $rvs(H)$, where H are any graph with pendant vertex, sunlet graph ($\mathcal{S}_n$), helm graph (H_n), subdivided star graph (SS_n), and broom graph ($Br_{n,m}$). They also determined the lower bound lemma of any graph H as follows.

Lemma 1.3. *[11] For any graph H of order p, δ, and Δ,*

$$rvs(H) \geq \left\lceil \frac{p+\delta-1}{\Delta+1} \right\rceil,$$

where δ and Δ are minimum and maximum degree of graph H, respectively.

In this paper, we will determine the exact value of some almost regular graphs. This type of graph firstly appeared in 1983 in Alon, Friendland, and Kalai paper, see [14]. They defined almost regular graph as a graph which all vertices degree are either t or $t+1$ and at least one vertex has each degree. The type of graphs considered as almost regular graph are ladder and complete bipartite $K_{2,n}$.

2 The Vertex Irregular Reflexive k-Labeling on Some Almost Regular Graph

The two almost regular graphs studied in this paper are ladder and bipartite complete $(K_{2,n})$ graphs. In the following theorem determine the $rvs(H)$, where H are ladder graph and bipartite complete $(K_{2,n})$.

Theorem 2.1. *Let L_n be a ladder graph and $n \geq 3$. The reflexive vertex strength of L_n is*

$$rvs(L_n) = \begin{cases} \left\lceil \dfrac{2n+1}{4} \right\rceil + 1 & \text{if } n \equiv 1 (mod 4) \\ \\ \left\lceil \dfrac{2n+1}{4} \right\rceil, & \text{otherwise} \end{cases}$$

Proof. A ladder graph, denoted by L_n has a vertex set $V(L_n) = \{x_l : l \in \{1,2,3,\ldots,n\}\} \cup \{y_l : l \in \{1,2,3,\ldots,n\}\}, |V(L_n)| = 2n$ and edge set $E(L_n) = \{x_l x_{l+1} : l \in \{1,2,3,\ldots,n-1\}, \cup \{y_l y_{l+1} : l \in \{1,2,3,\ldots,n-1\}, \cup \{x_l y_l : l \in \{1,2,3,\ldots,n\}\}, |E(L_n)| = 3n-2$. Refer to Lemma 1.3, We determine the following lower bound of $rvs(L_n)$.

$$rvs(L_n) \geq \left\lceil \frac{2n+1}{4} \right\rceil$$

For $n \equiv 1(mod\ 4)$, based on Lemma 1.1 which have discussed the largest vertex weight, we have

$$rvs(L_n) \geq \left\lceil \frac{p+\delta}{\Delta+1} \right\rceil = \left\lceil \frac{2n+2}{4} \right\rceil = \left\lceil \frac{2n+1}{4} \right\rceil + \left\lceil \frac{1}{4} \right\rceil = \left\lceil \frac{2n+1}{4} \right\rceil + 1$$

We determine the upper bound of $rvs(L_n)$ by using the following function of domain vertex and edge set. We use an illustration in Figure to show the labeling and determine the upper bound of $L_n, n \in \{3,4,5\}$.

Furthermore for $n \geq 6$, we define

$$k = \begin{cases} \left\lceil \frac{2n+1}{4} \right\rceil + 1 & \text{if } n \equiv 1(mod 4) \text{ and } n \geq 9 \\ \left\lceil \frac{2n+1}{4} \right\rceil, & \text{if } n = 5 \text{ and otherwise.} \end{cases}$$

$$f(x_l) = \begin{cases} 0, & \text{if } l \in \{1,2,3,\ldots,k+1\} \\ 2\left\lceil \frac{l-(k+1)}{2} \right\rceil, & \text{if } l \in \{k+2,k+3,k+4,\ldots,n\} \end{cases}$$

$$f(y_1) = f(x_n) + 2, f(y_n) = 2$$

To construct the labelings of $y_l, x_l x_{l+1}, y_l y_{l+1}, x_l y_l$, we divide into 4 cases.

Case 1. For $n \equiv 2(mod 4)$, we have

$$f(y_l) = 4 : l \in \{1,2,3,\ldots,n\}, n = 6$$

$$f(y_l) = \begin{cases} 4, & \text{if } l \in \{2,3,4,\ldots,k+1\} \\ 2\left\lceil \frac{l-(k+1)}{2} \right\rceil + 4, & \text{if } l \in \{k+2,k+3,k+4,\ldots,n-1\}\ n \geq 7 \end{cases}$$

$$g(x_l x_{l+1}) = g(x_1 y_1) = 1$$

$$g(x_l y_l) = \begin{cases} l-1, & \text{if } l \in \{2,3,4,\ldots,k+1\} \\ k-1, & \text{if } l \in \{k+2,k+3,k+4,\ldots,n-1\} \\ \text{and } l & \text{is even} \\ k, & \text{if } l \in \{k+3,k+4,k+5,\ldots,n-1\} \\ \text{and } l & \text{is odd and } l = n \end{cases}$$

$$g(y_l y_{l+1)} = k-1$$

We can determine the vertex weight on x_l and y_l from the vertex and edge labelings as follows.

$$W(x_l) = l+1, \text{if } l \in \{1,2,3,\ldots,n\}$$

$$W(y_l) = \begin{cases} n+2, & \text{if } l=1 \\ n+3, & \text{if } l=n \\ n+l+3, & \text{if } l \in \{2,3,4,\dots,n-1\} \end{cases}$$

Case 2. For $n \equiv 3(mod4)$, we have

$$f(y_l) = \begin{cases} 2, & \text{if } l \in \{2,3,4,\dots,k+1\} \\ 2\left\lceil \frac{l-(k+1)}{2} \right\rceil + 2, & \text{if } l \in \{k+2,k+3,k+4,\dots,n-1\} \end{cases}$$

$$g(x_l x_{l+1}) = g(x_1 y_1) = 1$$

$$g(x_l y_l) = \begin{cases} l-1, & \text{if } l \in \{2,3,4,\dots,k+1\} \\ k-1, & \text{if } l \in \{k+2,k+3,k+4,\dots,n-2\} \\ \text{and } l & \text{is even} \\ k, & \text{if } l \in \{k+3,k+4,k+5,\dots,n-2\} \\ \text{and } l & \text{is odd and } l=n-1,n \end{cases}$$

$$g(y_l y_{l+1)} = k$$

We can determine the vertex weight on x_l and y_l from the vertex and edge labelings as follows.

$$W(x_l) = \begin{cases} l+1, & \text{if } l \in \{1,2,3,\dots,n-2\} \\ n, & \text{if } l=n \\ n+1, & \text{if } l=n-1 \end{cases}$$

$$W(y_l) = \begin{cases} n+2, & \text{if } l=1 \\ n+3, & \text{if } l=n \\ n+l+2, & \text{if } l \in \{2,3,4,\dots,n-2\} \\ 2n+2, & \text{if } l=n-1 \end{cases}$$

Case 3. For $n \equiv 0(mod4)$, we have

$$f(y_l) = \begin{cases} 2, & \text{if } l \in \{2,3,4,\dots,k+1\} \\ 2\left\lceil \frac{l-(k+1)}{2} \right\rceil + 2, & \text{if } l \in \{k+2,k+3,k+4,\dots,n-1\} \end{cases}$$

$$g(x_i x_{i+1}) = g(x_1 y_1) = 1$$

$$g(x_ly_l) = \begin{cases} l-1, & \text{if } l \in \{2,3,4,\dots,k+1\} \\ k-1, & \text{if } l \in \{k+2,k+3,k+4,\dots,n-2\} \text{ and } l \text{ is odd} \\ k, & \text{if } l \in \{k+3,k+4,k+5,\dots,n-2\} \text{ and } l \text{ is even and } l = \{n-1,n\} \end{cases}$$

$$g(y_ly_{l+1)} = k, \text{if } l \in \{1,2,3,\dots,n-2\}$$

$$g(y_{n-1}y_n) = k-1$$

We can determine the vertex weight on x_l and y_l from the vertex and edge labelings as follows.

$$W(x_l) = \begin{cases} l+1, & \text{if } l \in \{1,2,3,\dots,n-2\} \\ n, & \text{if } l = n \\ n+1, & \text{if } l = n-1 \end{cases}$$

$$W(y_l) = \begin{cases} n+2, & \text{if } l = 1 \\ n+3, & \text{if } l = n \\ n+l+3, & \text{if } l \in \{2,3,4,\dots,n-1\} \end{cases}$$

Case 4. For $n \equiv 1(mod 4)$, we have

$$f(y_l) = \begin{cases} 0, & \text{if } l \in \{2,3,4,\dots,k+1\} \\ 2\left\lceil \frac{l-(k+1)}{2} \right\rceil, & \text{if } l \in \{k+2,k+3,k+4,\dots,n-1\} \end{cases}$$

$$g(x_lx_{l+1}) = g(x_1y_1) = 1$$

$$g(x_ly_l) = \begin{cases} l-1, & \text{if } l \in \{2,3,4,\dots,k+1\} \\ k-1, & \text{if } l \in \{k+2,k+3,k+4,\dots,n-2\} \\ \text{and } l & \text{is even} \\ k, & \text{if } l \in \{k+3,k+4,k+5,\dots,n-2\} \\ \text{and } l & \text{is odd and } l = n-1,n \end{cases}$$

$$g(y_ly_{l+1)} = k$$

We can determine the vertex weight on x_l and y_l from the vertex and edge labelings as follows.

$$W(x_l) = \begin{cases} l+1, & \text{if } l \in \{1,2,3,\dots,n-2\} \\ n, & \text{if } l = n \\ n+1, & \text{if } l = n-1 \end{cases}$$

$$W(y_l) = \begin{cases} n+2, & \text{if } l = 1 \\ n+3, & \text{if } l = n \\ n+l+2, & \text{if } l \in \{2,3,4,\dots,n-2\} \\ 2n+2, & \text{if } l = n-1 \end{cases}$$

We can see that the all the elements of $W(x_l)$ and $W(y_l)$ are all different. It completes the proof. □

Secondly, we will show our result on $rvs(H)$, where H is the complete bipartite graph in the following theorem.

Theorem 2.2. *Let $K_{2,n}$ be a complete bipartite graph and $n \geq 3$. The reflexive vertex strength of graph $K_{2,n}$ is*

$$rvs(K_{2,n}) = \begin{cases} \left\lceil \frac{n+1}{3} \right\rceil, & \text{if } n \equiv 0,1 (mod 4) \\ \left\lceil \frac{n+2}{3} \right\rceil, & \text{if } n \equiv 2,3 (mod 4) \end{cases}$$

Proof. A bipartite complete graph, denoted by $K_{2,n}$ has a vertex set $V(K_{2,n}) = \{x_1, x_2, y_l : l \in \{1,2,3,\dots,n\}\}$, $|V(K_{2,n})| = n+2$ and edge set $E(K_{2,n}) = \{x_1y_l, x_2y_l : l \in \{1,2,3,\dots,n\}\}$, $|E(K_{2,n})| = 2n$. The bipartite complete graph $K_{2,n}$ has 2 vertices of degree n. Since $n \geq 3$, the degree of vertex x_1 and x_2 are greater than other vertices, such that the vertex weight of x_1 and x_2 are always different from the others. Thus, we exclude the two vertices x_1 and x_2 to determine the lower bound of $rvs(K_{2,n})$. Hence, We refer to Corollary 1.2 to determine the lower bound of $rvs(K_{2,n})$ as follows.

$$rvs(K_{2,n}) \geq \begin{cases} \left\lceil \frac{n+1}{3} \right\rceil, & \text{if } n \equiv 0,1 (mod 4) \\ \left\lceil \frac{n+2}{3} \right\rceil, & \text{if } n \equiv 2,3 (mod 4) \end{cases}$$

Furthermore, we determine the upper bound of $rvs(K_{2,n})$ by using the following function of domain vertex and edge set. Define k as follows.

$$k = \begin{cases} \left\lceil \frac{n+1}{3} \right\rceil, & \text{if } n \equiv 0,1 (mod 4) \\ \left\lceil \frac{n+2}{3} \right\rceil, & \text{if } n \equiv 2,3 (mod 4) \end{cases}$$

$$f(x_1) = f(x_2) = \begin{cases} k-1, & \text{if } k \text{ odd} \\ k, & \text{if } k \text{ even} \end{cases}$$

For y_i, x_1y_i and x_2y_i labels, we present in Table 1 as follows.
Based on the Table 1, the vertex weight of x_1 and x_2 are

$$W(x_1) = \sum_{l=1}^{n} (x_1y_l)$$

	y_i	x_1y_i	x_2y_i	$W(y_i)$
y_1	0	1	1	2
y_2	0	1	2	3
y_3	2	1	1	4
y_4	2	1	2	5
y_5	2	2	2	6
y_6	2	2	3	7
y_7	2	3	3	8
y_8	4	2	3	9
y_9	4	3	3	10
y_{10}	4	3	4	11
y_{11}	4	4	4	12
y_{12}	4	4	5	13
y_{13}	4	5	5	14
y_{14}	6	4	5	15
y_{15}	6	5	5	16
y_{16}	6	5	6	17
y_{17}	6	6	6	18
y_{18}	6	6	7	19
y_{19}	6	7	7	20
y_{20}	8	6	7	21
y_{21}	8	7	7	22
y_{22}	8	7	8	23
y_{23}	8	8	8	24
y_{24}	8	8	9	25
y_{25}	8	9	9	26
...	...	...	...	...

Table 1. Labeling of vertex y_i, edges x_1y_i, x_2y_i, and $W(y_i)$

$$W(x_2) = \sum_{l=1}^{n}(x_2y_l)$$

We can see that the all the elements of $W(y_i), W(x_1)$ and $W(x_2)$ are all different. It completes the proof. □

We can see the illustration of the vertex irregular reflexive labeling of $K_{2,10}$ of in Figure 1. We can also determine the $rvs(K_{2,10})$ from those labelings.

3 Concluding Remarks

We have determined the exact value of reflexive vertex strength on some almost regular graph, namely ladder and bipartite complete graph $K_{2,n}$. To Determine the $rvs(H)$, where H is any graph family, is considered to be an NP-problem. Therefore we propose the following two open problems.

(i) Determine the $rvs(H)$, where H are any regular or almost regular graphs apart from the results obtained in this paper.

(ii) Determine the upper bound of $rvs(H)$, where H are any graphs.

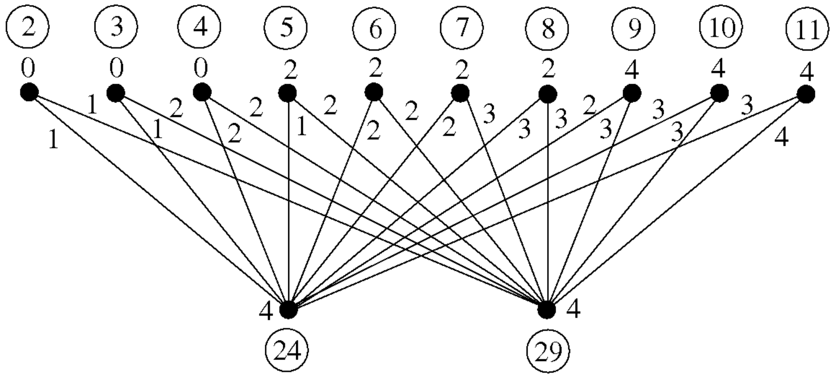

Figure 1. The Vertex Irregular Reflexive 4-labeling on $K_{2,10}$

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