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JEMES AND BIRKHOFF ORTHOGONALITY IN B(H)

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Abstract The general problem in this paper is minimizing the B(H)- norm of suitable affine mappings from B(H) to B(H), using convex and differential analysis (Gateaux derivative) as well as input from operator theory. The mappings considered generalize the so-called elementary operators and in particular the generalized derivations, which are of great interest by themselves. The main results obtained characterize global minima in terms of (Banach space)orthogonality

1 Introduction

The general problem in this paper is minimizing the B(H) norm of suitable affine mappings from B(H) to B(H) , using convex and differential analysis (Gateaux derivative) as well as input from operator theory. The mappings considered generalize the so-called elementary operators and in particular the generalized derivations, which are of great interest by themselves. The main results obtained characterize global minima in terms of (Banach space) orthogonality, and constitute an interesting combination of infinite-dimensional differential analysis, operator theory and duality. This leads us to characterize the orthogonality in the sense of Birkhoff in B(H) . Let B(H) be a complex Banach space. We first define orthogonality in $B(H)$. We say that $y \in B(H)$is orthogonal to $x \in B(H)$ if for all complex λ there holds,

$$\|x + \lambda y\|_{B(H)} \geq \|x\|_{B(H)} . \tag{1.1}$$

This definition has a natural geometric interpretation. Namely, $y \perp x$ if and only if the complex line $\{x + \lambda y | \lambda \in \mathbb{C}$ is orthogonal in $B(H)$, i.e, if and only if this complex line is a tangent one.

Definition 1.1. Let $B(H)$ be a Banach space, and $x, y \in B(H)$, x is a smooth point of the boundary of K in $B(H)$ if there exists a unique functional F_x, called the support functional, such that $\|F_x\| = 1$ and F_x=$\|x\|$

Remark 1.2. If $B(H)$ is a Hilbert space, from (1.1) we can easily derive $\langle x, y\rangle = 0$, i.e, orthogonality in the usual sense. In general, such orthogonality is not symmetric in Banach spaces. We can take as example the following vectors $(-1, 0)$ and $(1, 1)$, which are in the Hilbert-Schmidt classes C_2, with the max-norm. Joel Anderson has proved that, every non-zero $x \in B(H)$, is a smooth point if and only if $x \in B(H)$ attains its norm, $e \in H, \|xe\| = \|x\|$, and in this case the support functional of x is given by

$$D_T(x) = Retr\left[\frac{(e \otimes Te)}{\|T\|}x\right] = Re\left\langle xe, \frac{Te}{\|T\|}\right\rangle, \forall x \in B(H). \tag{1.2}$$

Here, Re denotes the real part and $D_T(X)$is the unique support functional (in the dual space $B(H)$ *), Recall that the rank one operator, $e \otimes Te$ is defined by

$$(e \otimes Te)x = \langle x, Te\rangle\, e,\ \forall x \in B(H). \tag{1.3}$$

and

$$tr(\frac{(e \otimes Te)}{\|T\|}x) = \left\langle xe, \frac{Te}{\|T\|}\right\rangle,\ \forall x \in B(H). \tag{1.4}$$

The first result concerning the orthogonality in Banach space was given by Anderson [2] ,showing that if A is a normal operator on a Hilbert space H and $S \in B(H)$ then $AS = SA$ implies that for any bounded linear operator x there holds

$$\|S + AX - XA\| \geq \|S\|. \tag{1.5}$$

This means that the range of the derivation

$$\delta_A : B(H) \to B(H),$$

defined by

$$\delta_A(X) = AX - XA, \tag{1.6}$$

is orthogonal to its kernel. This result has been generalized in two directions: by extending to the class of elementary mappings

$$E : B(H) \to B(H).$$

$$E(X) = \sum_{i=1}^{i=n} A_i X B_i,$$

and

$$\tilde{E} : B(H) \to B(H).$$

$$\tilde{E}(X) = \sum_{i=1}^{i=n} A_i X B_i - X,$$

where $(A_1, A_2, ...A_n)$ and$(B_1, B_2, ...B_n)$ are $n-$ tuples of bounded operators on H, and by extending the equality (1.2) to Cp the Schatten p-classes with $1 < p < \infty$ see [3], [16].The Gâteaux derivative concept was used in [4, 5, 11] and [6]. In order to characterize those operators which are orthogonal to the range of a derivation in C_p. First we characterize the global minimum of the map

$$X \to \|S + \Phi(X)\|,$$

were Φ is a linear map in $B(H)$, by using the Gâteaux derivative. These results are then applied to characterize the operators $S \in B(H)$ which are orthogonal to the range of elementary operators.

2 Preliminary

Proposition 2.1. *1) Let $B(H)$ be a Banach space, $x, y \in B(H)$, and $\varphi \in [0, 2\pi)$. The function*

$$\gamma : IR \to IR,$$

$$\gamma(t) = \left\|x + e^{it} y\right\|, \tag{2.1}$$

is convex.

The limit

$$D_{\varphi,x}(y) = \lim_{t \to 0^+} \frac{\left\|x + te^{i\varphi} y\right\| - \|x\|}{t}, \tag{2.2}$$

always exists.The number $D_{\varphi,x}(y)$ we shall call the φ-Gateaux derivative of the norm at the vector x, in the y and φ directions.

2)The vector y is orthogonal to x in the sense of James if and only if the inequality

$$\underset{\varphi}{Inf} D_{\varphi,x}(y) \geq 0, \tag{2.3}$$

holds.

Theorem 2.2. *Let $(B(H))$ be an arbitrary Banach space we define the function F: $B(H) \to IR$ $F(x) = \|x\|$. If F has a global minimum at $x \in B(H)$, then*

$$D_{F(x)}(y) \geq 0, \forall y \in B(H). \tag{2.4}$$

3 Main Results

Let φ be a linear map:

$$\varphi : B(H) \to B(H),$$

and let the map ψ defined by

$$\psi(x) = \varphi(x) + S, \tag{3.1}$$

for some element $S \in B(H)$.

$$D_x(y) = \lim_{t \to 0^+} \frac{\|x + ty\| - \|x\|}{t}, \tag{3.2}$$

such as

$$D_x(y) \leq \|y\|,$$

$$D_x(x) = \|x\|, \tag{3.3}$$

$$D_x(-x) = -\|x\| \tag{3.4}$$

Theorem 3.1. *The map $F_\psi(x) = \|\psi(x)\|$ has a global minimum at $x \in B(H)$ if and only if*

$$D_{\psi(x)}(\varphi(y)) \geq 0, \forall\, y \in B(H), \tag{3.5}$$

it is clear to see that

$$\psi(x) + t\varphi(y) = \psi(x + ty), \tag{3.6}$$

we choose t such that

$$\varphi(y - x) = \psi(y) - \psi(x), \tag{3.7}$$

$$D_{\psi(x)} = L,$$

then

$$\|\psi(x)\| = -L(-\psi(x)) \leq -(L(-\psi(x)) + L(\psi(y) - \psi(y)), \tag{3.8}$$

from where

$$\|\psi(x)\| \leq L(\psi(y)), \tag{3.9}$$

and by sub additivity we get

$$\|\psi(x)\| \leq \|\psi(y)\|. \tag{3.10}$$

In the following theorem we characterize the global minimum of the map F_ψ on $B(H)$ at V, when φ is a linear map

Theorem 3.2. *Let $V \in B(H)$ be a smooth point and f is a unique vector for which V attains its norm, then the map F_ψ has a global minimum at $V \in B(H)$,if and only if*

$$tr((f \otimes V)\varphi(y)) = 0, \forall\, y \in B(H). \tag{3.11}$$

Proof. Let $V \in B(H)$ be a smooth point
F_ψ has a global minimum on $B(H)$ at V, then

$$D_{\psi(V)}(\varphi(y)) \geq 0, \forall\, y \in B(H). \tag{3.12}$$

By (1.2) we get

$$Re(\langle \varphi(y)f, Vf \rangle) \geq 0, \forall\, y \in B(H). \tag{3.13}$$

Let Γ is the subspace of $B(H)$ in which $V \in B(H)$ attains its norm the set

$$\{\langle V^*\varphi(y)f, f\rangle \,/ f \in \Gamma, \|f\| = 1\}. \tag{3.14}$$

is numerical range of $V^*\varphi(y)$ on the subspace Γ. is convex and closed. By (3.5), it must contain a value whose real part is positive, under all rotations around the origin, it must contain the origin, and we will have a vectors $f \in \Gamma$ such that

$$\langle V^*\varphi(y)f, f\rangle < \delta.\ forall\ y \in B(H). \tag{3.15}$$

Where $\delta > 0$, as δ is arbitrary we can easily check that

$$\langle V^* \varphi(y) f, f \rangle = 0, \forall\, y \in B(H). \tag{3.16}$$

Then

$$tr((f \otimes V)\varphi(y)) = 0, \forall\, y \in B(H). \tag{3.17}$$

ii) Suppose that

$$tr((f \otimes V)\varphi(y)) = 0, \forall\, y \in B(H). \tag{3.18}$$

Then we use the arguments of least proof i) we get

$$Re(\langle \varphi(y) f, V f \rangle) \geq 0, \forall\, y \in B(H), \tag{3.19}$$

which completes the proof of the second part of the theorem. □

Let $\varphi = \delta_{A,B}$.

$$\delta_{A,B} : B(H) \to B(H),$$

is the generalized derivation defined by

$$\delta_{A,B}(X) = AX - XB. \tag{3.20}$$

Corollary 3.3. *Let $V \in B(H)$ be a smooth point, and f is the unitary vector in which $V \in B(H)$ attains its norm, then F_ψ has a global minimum at $V \in B(H)$,if and only if*

$$f \otimes \psi(V) f \in Ker\delta_{B,A}. \tag{3.21}$$

Proof. It is easily seen that

$$f \otimes \psi(V) f \in Ker\delta_{A,B} \Leftrightarrow tr((f \otimes V)\delta_{A,B} = 0. \tag{3.22}$$

□

Theorem 3.4. *Let $S \in B(H)$ be a smooth point, then*

$$\|S + (AX - XB)\|_{B(H)} \geq \|\psi(S)\|_{B(H)}, \forall X \in B(H), \tag{3.23}$$

if and only if $\exists f \in \Gamma, \|f\| = 1$, such that

$$f \otimes \psi(S) f \in Ker\delta_{B,A} \tag{3.24}$$

Proof. This theorem is a particular case of the previous one. its proof is trivial. □

Corollary 3.5. *Let $V \in B(H)$ be a smooth point, f is the unitary vector in which $V \in B(H)$ attains its norm, if $S \in Ker\delta_{A,B}$, then the following assertions are equivalent*
1)

$$\|S + (AX - XB)\|_{B(H)} \geq \|\psi(S)\|_{B(H)}, \forall X \in B(H). \tag{3.25}$$

2)

$$f \otimes \psi(S) f \in Ker\delta_{B,A}. \tag{3.26}$$

Remark 3.6. We point out that, thanks to our general results given previously with more general linear maps ψ.Theorem 3.4 and its Corollary 3.3 are still true for more general classes of operators than $\delta_{A,B}$ such as the elementary operators $E(X)$ and $\tilde{E}(X)$. Note that Theorem 3.4 and Corollary 3.5 generalize the results given in [8].

us some applications of the previous corollary let $\Delta_{A,B}$ the elementary operator defined by

$$\Delta_{A,B} : B(H) \to B(H)$$

$$\Delta_{A,B}(X) = AXB - X. \tag{3.27}$$

Theorem 3.7. *Let $S \in B(H)$ be a smooth point, and $A, B \in B(H)$,are contractions,that verify*

$$\Delta_{A,B}(S) = 0. \tag{3.28}$$

Then $\exists\, \tilde{S} \in B(H)$, verify

$$\Delta_{A,B}(\tilde{S}) = 0 = \Delta_{A^*,B^*}(\tilde{S}) \tag{3.29}$$

Proof. if $A, B \in B(H)$,are contractions, that verify

$$\Delta_{A,B}(S) = 0. \tag{3.30}$$

□

$$\left\|\Delta_{A,B}(\tilde{S}) + S\right\|_{B(H)} \geq \|S\|_{B(H)}, \forall \tilde{S} \in B(H). \tag{3.31}$$

Suppose that $S \in B(H)$ be a smooth point,we use the Corollary 3.5 applied to $\Delta_{A,B}$, $\exists\, f \in H$, hold's

$$\Delta_{A,B}(f \otimes Sf) = 0 = \Delta_{A^*,B^*}(f \otimes Sf), \tag{3.32}$$

taking

$$\tilde{S} = f \otimes Sf. \tag{3.33}$$

completes the proof. On the other hand, we applied the Corollary 3.5 we obtain

$$\Delta_{A,B}(\tilde{S}) = 0 = \Delta_{A,B}(s) \Leftrightarrow \left\|\Delta_{A,B}(\tilde{S}) + S\right\|_{B(H)} \geq \|S\|_{B(H)}, \forall\, S \in B(H). \tag{3.34}$$

Now we will present an other characterization of the orthogonality in the sense of Birkhoff.

Theorem 3.8. *Let $S \in B(H)$ be a smooth point, and $Y \in B(H)$, then the following assertions are equivalent.*

(i) The map F_ψ has a global minimum at $S \in B(H)$.

(ii) There exists unitary vector $f \in \Gamma$, such that

$$Re(< (\varphi(Y)f.Sf) >) \geq 0. \tag{3.35}$$

(iii) There exists unitary vector $f \in \Gamma$, such that

$$tr((f \otimes Sf)\varphi(Y)) = 0, \forall\, Y \in B(H). \tag{3.36}$$

(iv) there exists unitary vectors $f_n \in \Gamma$, such that

$$\|Sf_n\|_{B(H)} \underset{n\to+\infty}{\to} \|S\|_{B(H)} \tag{3.37}$$

and

$$\langle \varphi(Y)f_n, Sf_n\rangle \underset{n\to+\infty}{\to} 0 \tag{3.38}$$

Proof.

$$(i) \Rightarrow (ii)$$

We use the Theorem 3.1 and Theorem 3.2

$$(ii) \Leftrightarrow (iii)$$

see Theorem 3.2

$$(iv) \Rightarrow (i)$$

Its easily to see that $X \perp Y \in B(H)$, in the sense of Birkhoff if and only if

$$D_X(Y) \geq 0, \forall\, Y \in B(H). \tag{3.39}$$

□

We prove that

$$\|S+\lambda\varphi(Y)\|_{B(H)} \geq \|S\|_{B(H)}, \forall\, \lambda \in \mathbb{C}. \tag{3.40}$$

There exists a sequence of unitary vectors $f_n \in \Gamma$, such that

$$\|Sf_n\|_{B(H)} \underset{n\to+\infty}{\to} \|S\|_{B(H)}, \tag{3.41}$$

$$\langle \varphi(Y)f_n, Sf_n\rangle \underset{n\to+\infty}{\to} 0. \tag{3.42}$$

Then

$$\|S+\lambda\varphi(Y)\|^2_{B(H)} \geq \|S+\lambda\varphi(Y)f_n\|^2_{B(H)} \tag{3.43}$$

$$\geq \|Sf_n\|^2_{B(H)} + 2Re\lambda < \varphi(Y)f_n.Sf_n > + \|\varphi(Y)f_n\|^2_{B(H)}$$

$$\geq \|Sf_n\|^2_{B(H)} + 2Re(< \varphi(Y)f_n.Sf_n >) \underset{n\to+\infty}{\to} \|S\|^2_{B(H)}. \tag{3.44}$$

$$(iii) \Rightarrow (iv)$$

On the other hand, in the case of the proof of Theorem 3.2 we obtain unitary vector f such that

$$|< \varphi(Y)f.Sf >| < \delta. \tag{3.45}$$

let $N \in \mathbb{N}^*$, if we take $\delta \to \frac{1}{N}$ we get the result

Corollary 3.9. *Let $\varphi(Y) = \delta_{A,B}(Y) = AY - YB$, and $S, Y \in B(H)$, where S is a smooth point, then the following conditions are equivalent.*
1) The map $\|S + AY - YB\|^2_{B(H)}$ has a global minimum at $S \in B(H)$
2) There exist unitary vector $f \in \Gamma$, such that

$$Re\,\langle (AY - YB)f, Sf\rangle \geq 0. \tag{3.46}$$

3) There exist unitary vector $f \in \Gamma$, such that

$$tr((f \otimes Sf)(AY - YB) = 0, \forall\, Y \in B(H), \tag{3.47}$$

4) There exists a sequence of unitary vectors $f_n \in \Gamma$, such that

$$\|Sf_n\|_{B(H)} \underset{n\to+\infty}{\to} \|S\|_{B(H)} \tag{3.48}$$

and

$$\langle (AY - YB)f_n, Sf_n\rangle \underset{n\to+\infty}{\to} 0. \tag{3.49}$$

If $S \in Ker\delta_{A,B}$, we obtain the following corollary

Corollary 3.10. *Let $\varphi(Y) = \delta_{A,B}(Y) = AY - YB$, and $S, Y \in B(H)$, where S is a smooth point, then the following assertions are equivalent.*
1)

$$\|S + AY - YB\|^2_{B(H)} \geq \|S\|_{B(H)}, \forall S \in Ker\delta_{A,B}. \tag{3.50}$$

2) There exist unitary vector $f \in \Gamma$, such that

$$Re\,\langle (AY - YB)f.Sf\rangle \geq 0. \tag{3.51}$$

3) There exist unitary vector $f \in \Gamma$, such that

$$tr((f \otimes Sf)(AY - YB) = 0, \forall\, Y \in B(H). \tag{3.52}$$

4) There exists a sequence of unitary vectors $f_n \in \Gamma$, such that

$$\|Sf_n\|_{B(H)} \underset{n\to+\infty}{\to} \|S\|_{B(H)}, \tag{3.53}$$

and

$$\langle (AY - YB)f_n.Sf_n\rangle \underset{n\to+\infty}{\to} 0. \tag{3.54}$$

If we put

$$\varphi(Y) = Y.$$

Then we obtain the following corollary

Corollary 3.11. *Let $\varphi(Y) = Y$, and $S, Y \in B(H)$, where S is a smooth point, then the following conditions are equivalent.*
(1) $Y \perp S$, in the sense of Birkhoff.
(2) There exist unitary vector $f \in \Gamma$, such that

$$Re \langle Yf.Sf\rangle \geq 0. \tag{3.55}$$

(3) There exist unitary vector $f \in \Gamma$, such that

$$tr((f \otimes Sf)Y) = 0, \forall\, Y \in B(H). \tag{3.56}$$

(4) There exist a sequence of unitary vectors $f_n \in \Gamma$, such that

$$\|Sf_n\|_{B(H)} \underset{n\to+\infty}{\rightarrow} \|S\|_{B(H)}\,. \tag{3.57}$$

$$\langle (AY - YB)f_n.Sf_n\rangle \underset{n\to+\infty}{\rightarrow} 0 \tag{3.58}$$

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