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On some classes of modules related to chain conditions

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Abstract We discuss some variants of ascending and descending chain conditions (see [2], [3] and [8]) analogously. We introduce the idea of nem-Noetherian and nem-Artinian modules and rings. A right R-module M is said to be nem-Noetherian (nem-Artinian) if for every ascending (descending) chain $M_1 \leq M_2 \leq M_3 \leq \ldots (M_1 \geq M_2 \geq M_3 \geq \ldots)$ of non-essential submodules of M, there exists an index n such that M_{i+1} embeds in M_i (M_i embeds in M_{i+1}), for every $i \geq n$. We characterize these modules such that M is nem-Artinian (nem-Noetherian) if and only if every non-essential submodule of M is mono-Artinian (mono-Noetherian) if and only if every proper closed submodule of M is mono-Artinian (mono-Noetherian). Also, we study several properties of these modules.

1 Introduction

Let R denote a ring (associative with identity) and M a unitary right R-module. Noether and Artin studied the ascending chain conditions (ACC) and the descending chain conditions (DCC) on ideals in a ring (submodules in a module). Many authors realized the importance of these concepts and generalized them in various ways. In 2016, Facchini and Nazemian studied the modules with chain conditions upto isomorphism and discussed Artinian dimensions and iso-radical of modules in [6] and [5]. They call these modules by iso-Noetherian and iso-Artinian modules, respectively. Also, several characterizations and properties of these concepts have been investigated by them. Dastanpour et.al. [4] studied modules with epimorphism on any chain of submodules. They generalize the notion of iso-Noetherian and iso-Artinian modules and call them as epi-ACC and epi-DCC. In [8], we discuss some properties of iso-Noetherian and iso-Artinian rings and modules. We have studied the classes of mono-Noetherian and mono-Artinian modules in [2]. These notions are dual to that of epi-ACC and epi-DCC on submodules. We have studied some new variants of chain conditions.

Recall [7], two rings R, S are said to be Morita equivalent if there exists a category equivalence $F : M_R \to M_S$. A ring theoretic property P is said to be *Morita invariant* if, whenever R has the property P, so does every $S \approx R$. Recall, a submodule N of an R-module M is said to be *closed* if N has no proper essential extension in M and a submodule L of a module M is said to be essential submodule if $L \cap K \neq 0$, for every nonzero submodule K of M, otherwise L is non-essential. An R-module U is *uniform* provided $U \neq 0$ and $V \cap W \neq 0$ for all nonzero submodules V, W of U. Recall [7], an R-module M has finite *uniform dimension* (also known as Goldie dimension) if there is an essential submodule L of M that is a finite direct sum of uniform submodules.

In [3], we generalized the notion of iso-Noetherian (iso-Artinian) modules to nei-Noetherian (nei-Artinian) and ei-Noetherian (ei-Artinian) modules. In these notions, we consider the chain of non-essential and essential submodules, respectively. In the present work, we define the notions of nem-Noetherian (em-Noetherian) and nem-Artinian (em-Artinian) modules analogous to Definition 2.3 and Definition 2.4. We provide some characterizations and study many new properties of these classes of modules and rings. For undefined terms and notions, we refer [1] and [7].

2 Definitions and Examples

Definition 2.1. (See [6]) An R-module M is iso-Artinian if for every descending chain $M_1 \geq M_2 \geq M_3 \geq \ldots$ of submodules of M, there exists an index n such that M_n is isomorphic to M_i for every $i \geq n$. Dually, a module M is iso-Noetherian if for every ascending chain $M_1 \leq M_2 \leq M_3 \leq \ldots$ of submodules of M, there exists an index n such that M_n is isomorphic to M_i for every $i \geq n$. A ring R is said to be right iso-Noetherian (right iso-Artinian) if the right module R_R is iso-Noetherian (iso-Artinian).

Definition 2.2. [2, Definition 3.1] A right R-module M is *mono-Artinian* if for every descending chain $M_1 \geq M_2 \geq M_3 \geq \ldots$ of submodules of M, there exists $n \in \mathbb{N}$ such that M_i embeds in M_{i+1}, for all $i \geq n$. A ring R is right mono-Artinian if the right R-module R is mono-Artinian. A ring R is said to be mono-Artinian if it is both left as well as right mono-Artinian. Similarly, we can define mono-Noetherian modules and rings.

Definition 2.3. [3, Definition 2.1] A right R-module M is said to be nei-Noetherian (nei-Artinian) if for every ascending (descending) chain $M_1 \leq M_2 \leq M_3 \leq \ldots (M_1 \geq M_2 \geq M_3 \geq \ldots)$ of non-essential submodules of M, there exists an index n such that M_i is isomorphic to M_n, for every $i \geq n$.

Definition 2.4. [3, Definition 3.1] A right R-module M is said to be ei-Noetherian (ei-Artinian) if for every ascending (descending) chain $M_1 \leq M_2 \leq M_3 \leq \ldots (M_1 \geq M_2 \geq M_3 \geq \ldots)$ of essential submodules of M, there exists an index n such that M_i is isomorphic to M_n, for every $i \geq n$.

Definition 2.5. A right R-module M is said to be nem-Noetherian (nem-Artinian) if for every ascending (descending) chain $M_1 \leq M_2 \leq M_3 \leq \ldots (M_1 \geq M_2 \geq M_3 \geq \ldots)$ of non-essential submodules of M, there exists an index n such that M_{i+1} embeds in M_i (M_i embeds in M_{i+1}), for every $i \geq n$.

Definition 2.6. A right R-module M is said to be em-Noetherian (em-Artinian) if for every ascending (descending) chain $M_1 \leq M_2 \leq M_3 \leq \ldots (M_1 \geq M_2 \geq M_3 \geq \ldots)$ of essential submodules of M, there exists an index n such that M_{i+1} embeds in M_i (M_i embeds in M_{i+1}), for every $i \geq n$.

We observe the following implications, however the converse is not true in all cases.

$$\begin{array}{ccccc} Noetherian & \Longrightarrow & iso-Noetherian & \Longrightarrow & mono-Noetherian \\ & \searrow & \Downarrow & & \Downarrow \\ & & nei-Noetherian & \Longrightarrow & nem-Noetherian \end{array}$$

$$\begin{array}{ccccc} Noetherian & \Longrightarrow & iso-Noetherian & \Longrightarrow & mono-Noetherian \\ & \searrow & \Downarrow & & \Downarrow \\ & & ei-Noetherian & \Longrightarrow & em-Noetherian \end{array}$$

In order to sustain our assertion, we provide some examples and counter examples.

Example 2.7. (i) Every mono-Noetherian (mono-Artinian) module is em-Noetherian (em-Artinian). But a $\mathbb{Z}$-module $M = \oplus_{p_i \in \mathbb{P}} \mathbb{Z}_{p_i}$, where $\mathbb{P}$ is the set of prime integers is em-Noetherian (em-Artinian) but not mono-Noetherian (mono-Artinian). Also, it is not a nem-Noetherian (nem-Artinian) module.

(ii) Every mono-Noetherian (mono-Artinian) module is nem-Noetherian (nem-Artinian). However, every uniform module is nem-Noetherian (nem-Artinian) but it is not necessarily mono-Noetherian (mono-Artinian).

(iii) Semisimple modules are em-Noetherian (em-Artinian) but need not be iso-Noetherian (iso-Artinian).

(iv) A uniform module is mono-Noetherian if and only if it is em-Noetherian.

3 nem-Noetherian and nem-Artinian modules

In [3, Theorem 2.4 and Theorem 2.5], we have characterized nei-Artinian and nei-Noetherian modules. In the following result, we characterize nem-Artinian (nem-Noetherian) modules analogous to that.

Theorem 3.1. The following are equivalent for an R-module M:

(i) M is nem-Artinian (nem-Noetherian).

(ii) Every non-essential submodule of M is mono-Artinian (mono-Noetherian).

(iii) Every proper closed submodule of M is mono-Artinian (mono-Noetherian).

Proof. $(1) \Rightarrow (2)$. Let N be a non-essential submodule of M. Let $N_1 \geq N_2 \geq N_3 \geq \dots$ be a descending chain of submodules of N. Since N is a non-essential submodule of M, each N_i is non-essential in M. So, there exists $k \in \mathbb{N}$ such that N_i embeds in N_{i+1}, for all $i \geq k$. Thus, N is mono-Artinian.

$(2) \Rightarrow (3)$. It follows by the fact that every proper closed submodule of M is always non-essential.

$(3) \Rightarrow (2)$. Let N_1 be a non-essential submodule of M. If N_1 is a closed submodule of M, then we are done. If not, there exists a proper essential extension N of N_1 such that N is a proper closed submodule of M. If N is not proper then N_1 becomes essential in M. Consider a descending chain $N_1 \geq N_2 \geq \dots$ of submodules of N_1. Then we have a descending chain $N \geq N_1 \geq N_2 \geq \dots$ of submodules of N. Since N is mono-Artinian, there exists $n \in \mathbb{N}$ such that N_i embeds in N_{i+1}, for all $i \geq n$. Therefore, N_1 becomes mono-Artinian.

$(2) \Rightarrow (1)$. Let $M_1 \geq M_2 \geq \dots$ be a descending chain of non-essential submodules of M. Since M_1 is non-essential, it follows by the assumption that M_1 is mono-Artinian. Therefore, there exists $n \in \mathbb{N}$ such that M_i embeds in M_{i+1}, for all $i \geq n$. Thus, M is nem-Artinian. □

The proof of the following two results are on the same line to [2, Proposition 3.6], however we give the proof of one for completeness.

Proposition 3.2. Let $R = \prod_{i=1}^{n} R_i$. Then R is right em-Artinian (em-Noetherian) ring if and only if each R_i is a right em-Artinian (em-Noetherian) ring.

Proof. Let $R_1, R_2, \dots, R_n$ be right em-Artinian rings and $R = R_1 \times R_2 \times \dots \times R_n$. Let $I_1 \geq I_2 \geq I_3 \geq \dots$ be a descending chain of essential right ideals of R. For each $j \in \mathbb{N}$, I_j is of the form $I_j = A_{j1} \times A_{j2} \times \dots \times A_{jn}$, where each A_{jk} is an essential right ideal of R_j. Since for each $k \in \{1, 2, 3, \dots, n\}$, R_k is right em-Artinian, there exists $m \in \mathbb{N}$ such that for each $j \geq m$ there is a monomorphism, say $\psi_{jk} : A_{jk} \to A_{(j+1)k}$. For each $j \geq m$, we define a map $\psi_j : I_j \to I_{j+1}$ by $\psi_j(a_{j1}, a_{j2}, \dots, a_{jn}) = (\psi_{j1}(a_{j1}), \psi_{j2}(a_{j2}), \dots, \psi_{jn}(a_{jn}))$, for every $(a_{j1}, a_{j2}, \dots, a_{jn}) \in I_j$. Since each ψ_{jk} is a monomorphism, ψ_j is a monomorphism. Hence R is right em-Artinian. The converse is obvious. In case of right em-Noetherian, if we consider ascending chain of right ideals then the proof is on the same line. □

Proposition 3.3. Let $R = \prod_{i=1}^{n} R_i$, then R is em-Artinian (em-Noetherian) ring if and only if each R_i is an em-Artinian (em-Noetherian) ring.

We are unable to show that whether the above results hold true in case of the direct sum of modules. In the following, we raise some open problems and further give partial answers to them.

Question 3.4. (i) Is the direct sum of em-Artinian (em-Noetherian) modules em-Artinian (em-Noetherian)?

(ii) Is the direct sum of nem-Artinian (nem-Noetherian) modules nem-Artinian (nem-Noetherian)?

Proposition 3.5. Let $M = \oplus_{i=1}^{n} M_i$ be a direct sum of em-Noetherian (em-Artinian) modules M_i. If M is distributive, then M is em-Noetherian (em-Artinian).

Proof. We prove in case of ei-Noetherian. Let $L_1 \leq L_2 \leq \ldots$ be an ascending chain of essential submodules of M. For each $j \in \{1, 2, \ldots, n\}$, $L_i \cap M_j$ are essential submodules of M_j, $\forall\ i$. Thus, we have an ascending chain $L_1 \cap M_j \leq L_2 \cap M_j \leq \ldots$ of essential submodules of M_j. Since M_j is ei-Noetherian, there exists a positive integer $n_j \in \mathbb{N}$ such that $L_{n_j} \cap M_j \cong L_k \cap M_j$, for all $k \geq n_j$. Let $n = max\{n_1, n_2, \ldots n_n\}$. Then $L_n \cap M_j \cong L_k \cap M_j$, for all $k \geq n$. Let $f_{kj} : L_{k+1} \cap M_j \to L_k \cap M_j$ be isomorphism, for all j. Since M is distributive, $L_i = \oplus_{j=1}^{n}(L_i \cap M_j)$, for all i. Now the map $f = \sum_{j=1}^{n} f_{kj} : L_{k+1} \to L_k$ is an isomorphism. □

In the following, we generalize [6, Proposition 5.1] and its proof in case of nem-Noetherian module.

Theorem 3.6. If M is a nem-Noetherian R-module, then $u.dim(M) < \infty$.

Proof. Let M be a nem-Noetherian R-module. If possible, suppose uniform dimension of M is not finite. Let $\oplus_{i=1}^{\infty} M_i$ be a submodule of M, where each M_i is nonzero submodule of M. Let $0 \neq a_i \in M_i$, for each i. Then, $\oplus_{i=1}^{\infty} a_i R$ is a nem-Noetherian R-module. Consider the ascending chain $a_1R \leq a_1R \oplus a_2R \leq a_1R \oplus a_2R \oplus a_3R \leq \ldots$ of non-essential submodules of M. Then, there exists $n_1 \geq 1$ such that $a_1R \oplus a_2R \oplus \ldots \oplus a_{n_1}R \oplus a_{n_1+1}R$ embeds in $a_1R \oplus a_2R \oplus \ldots \oplus a_{n_1}R$. This implies that for some $1 \leq j \leq n_1$, a_jR is not Noetherian. For, if all $a_iR, 1 \leq i \leq n_1$ is Noetherian, then $u.dim(a_1R \oplus a_2R \oplus \ldots \oplus a_{n_1}R) \neq u.dim(a_1R \oplus a_2R \oplus \ldots \oplus a_{n_1}R \oplus a_{n_1+1}R)$, which is not possible. Set $b_1 = a_j$. Similarly, $\oplus_{i=n_1+1}^{\infty} a_iR$ is nem-Noetherian, so that there exists $j \geq n_1 + 1$ such that a_jR is not Noetherian. Set $b_2 = a_j$. Continuing in this manner, we find a sequence $b_1, b_2, \ldots$ such that $\oplus_{i=1}^{\infty} b_iR$ is nem-Noetherian and each b_iR is not Noetherian. For each $i \geq 1$, let K_i be a non-finitely generated submodule of b_iR. Consider the ascending chain $K_1 \leq b_1R \leq b_1R \oplus K_2 \leq b_1R \oplus b_2R \leq b_1R \oplus b_2R \oplus K_3 \leq \ldots$ of submodules of $\oplus_{i=1}^{\infty} b_iR$. There exists $n \geq 1$ such that $b_1R \oplus b_2R \oplus \ldots b_nR \oplus K_{n+1} \cong b_1R \oplus b_2R \oplus \ldots b_nR \oplus b_{n+1}R$. This implies that K_{n+1} is finitely generated, which is a contradiction. □

Corollary 3.7. Every mono-Noetherian module has finite Goldie dimension.

Proposition 3.8. Being nem-Noetherian, nem-Artinian, em-Noetherian, em-Artinian, nei-Noetherian and ei-Noetherian modules are a Morita invariant property.

Recall [9], a nonzero R-module M is called *compressible* if for each nonzero submodule N of M there exists a monomorphism $f : M \to N$.

Lemma 3.9. Every nem-Artinian module contains a compressible module.

Proof. Let M be a nem-Artinian module and $M_1 \geq M_2 \geq M_3 \geq \ldots$ be a descending chain of non-essential submodules of M. Then, there exists $n \in \mathbb{N}$ such that $M_i \cong M_n$, for all $i \geq n$. Clearly, M_n is non-essential and by Theorem 3.1, M_n is mono-Artinian. Therefore, M_n contains a compressible submodule. □

Recall, an ideal I of R is *semiprime* if $A^2 \subseteq I$ implies $A \subseteq I$ for any ideal A of R. A ring R is semiprime if (0) is a semiprime ideal of R.

Proposition 3.10. The following are equivalent for a right nem-Artinian ring R:

(i) R is semiprime.

(ii) The intersection of the annihilator of compressible right R-modules is zero.

Proof. $(1) \Rightarrow (2)$. Let I be the intersection of the annihilator of all the compressible right R-modules. Suppose $I = 0$. Let J be an ideal of R such that $J^2 = 0$. Then $MJ = 0$, for all compressible right R-modules M. For otherwise, if $MJ \neq 0$ for some compressible right R-module M, then M embeds in MJ implies that $MJ^2 = MJ = 0$, which is a contradiction. Thus, $J \leq I = 0$ i.e. $J = 0$. Therefore, R is semiprime.

$(2) \Rightarrow (1)$. Let R be a semiprime ring. If possible, suppose $I \neq 0$, then I is nem-Artinian R-module and hence by Lemma 3.9, it contains an compressible right ideal C of R. Therefore, $C^2 = C.C = 0$ because C is compressible and $C \subseteq I$. Since R is semiprime, therefore $C = 0$, a contradiction. Thus, $I = 0$. □

It is always interesting to determine the endomorphism ring of a module or ring. We raise this problem for the following classes of modules and rings.

Question 3.11. How to determine the endomorphism rings of em-Noetherian, nem-Noetherian, em-Artinian, nem-Artinian rings and modules.

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