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CLOSED-FORM SOLUTIONS TO SOME NONLINEAR FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS ARISING IN MATHEMATICAL SCIENCES

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Abstract The purpose of this paper is to combine two powerful methods, the natural transform method and the reduced differential transform method to get a better method for solving nonlinear time-fractional partial differential equations. This method is called the natural reduced differential transform method (NRDTM). The time-fractional derivatives are taken in the Caputo sense. Three different numerical applications are given to demonstrate the efficiency and accuracy of the NRDTM. The obtained results show that the proposed method is very effective and easy to use for solving nonlinear fractional differential equations arising in mathematical sciences.

1 Introduction

Nonlinear fractional partial differential equations (NFPDEs) have played a very important role in various fields of science and engineering such as, mechanics, electricity, chemistry, biology, control theory, signal processing and image processing. In all of these scientific fields, it is important to obtain exact or approximate solutions of NFPDEs. But in general, there exists no method that gives an exact solution for NFPDEs and most of the obtained solutions are only approximations. Searching of exact solutions of NFPDEs in mathematical and other scientific applications is still quite challenging and needs new methods. Computing the exact solution of these equations is of considerable importance, because the exact solutions can help to understand the mechanism and complexity of phenomena that have been modeled by NFPDEs. [4, 7, 8, 10, 14, 16].

In the literature, there are many methods for solving NFPDEs. Among these methods: the Adomian decomposition method (ADM) [13], homotopy perturbation method (HPM) [3], homotopy analysis method (HAM) [1], generalized differential transform method (GDTM) [6], fractional variational iteration method (FVIM) [15], fractional reduced differential transform method (RDTM) [2], residual power series method (RPSM) [9].

The main objective of this paper is to propose a new iterative method for obtaining an analytical solution of nonlinear time-fractional partial differential equations. The new iterative method basically illustrates how two powerful methods can be combined and used to get exact solutions to nonlinear fractional partial differential equations arising in mathematical sciences.

The rest of this paper is organized as follows: In Section 2, we discuss the necessary definitions and mathematical preliminaries of the fractional calculus theory and the natural transform. In Section 3, we provide some important definitions and operations of the reduced differential transform method. In Section 4, we describe the fundamental idea of the NRDTM and in Section 5, we demonstrate the accuracy and effectiveness of the method by considering three cases of nonlinear time-fractional partial differential equations. Finally, we present the conclusion in Section 6.

2 Preliminaries

For the convenience of the reader, here we present the necessary definitions and properties of the fractional calculus theory and natural transform.

Definition 2.1. [12] Let $f(t) \in C[0,T]$ and $T > 0$. Then, the α order Riemann-Liouville fractional integral operator is given as

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{0}^{t} (t-s)^{\alpha-1} f(s) ds, \tag{2.1}$$

where $\Gamma(.)$ denotes the gamma function, $\alpha \geq 0$ and $t > 0$.

Definition 2.2. [12] Let $f^{(n)}(t) \in C[0,T]$ and $T > 0$. Then, the α order Caputo fractional derivative operator is given as

$$D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{0}^{t} (t-s)^{n-\alpha-1} f^{(n)}(s) ds, \tag{2.2}$$

where $n-1 < \alpha \leq n, n \in \mathbb{N}$ and $t > 0$.

For the Riemann-Liouville fractional integral and Caputo fractional derivative, we have the following relation

$$I^{\alpha} D^{\alpha} f(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(0^{+}) \frac{t^{k}}{k!}, \tag{2.3}$$

where $t > 0$.

Definition 2.3. [12] The Mittag-Leffler function is defined as follows

$$E_{\alpha}(z) = \sum_{i=0}^{\infty} \frac{z^{i}}{\Gamma(i\alpha+1)}, \alpha > 0, z \in \mathbb{C}. \tag{2.4}$$

A further generalization of (2.4) is given in the form

$$E_{\alpha,\beta}(z) = \sum_{i=0}^{\infty} \frac{z^{i}}{\Gamma(i\alpha+\beta)}, \alpha, \beta > 0, z \in \mathbb{C}. \tag{2.5}$$

For $\alpha = 1$, $E_{\alpha}(z)$ reduces to e^{z}.

Definition 2.4. [5] The natural transform is defined over the set of functions is defined over the set of functions

$$A = \left\{ f(t) / \exists M, \tau_1, \tau_2 > 0, |f(t)| < M e^{\frac{|t|}{\tau_j}} \text{ if } t \in (-1)^{j} \times [0, \infty), j \in \mathbb{Z}^{+} \right\},$$

by the following integral

$$\mathcal{N}^{+}[f(t)] = R^{+}(s,v) = \frac{1}{v} \int_{0}^{+\infty} e^{-\frac{st}{v}} f(t) dt, s, v \in (0, \infty). \tag{2.6}$$

Some basic properties of the natural transform are given as follows:
Property 1: The natural transform is a linear operator. That is, if λ and μ are non–zero constants, then

$$\mathcal{N}^{+}[\lambda f(t) \pm \mu g(t)] = \lambda \mathcal{N}^{+}[f(t)] \pm \mu \mathcal{N}^{+}[g(t)].$$

Property 2: If $f^{(n)}(t)$ is the n-th derivative of function $f(t)$ w.r.t. "t" then its natural transform is given by

$$\mathcal{N}^{+}\left[f^{(n)}(t)\right] = \frac{s^n}{v^n} R^{+}(s,v) - \sum_{k=0}^{n-1} \frac{s^{n-(k+1)}}{v^{n-k}} f^{(k)}(0).$$

Property 3: (Convolution property) Suppose $F^{+}(s,v)$ and $G^{+}(s,v)$ are the natural transforms of $f(t)$ and $g(t)$, respectively, both defined in the set A. Then the natural transform of their convolution is given by

$$\mathcal{N}^{+}\left[(f*g)(t)\right] = vF^{+}(s,v)G^{+}(s,v),$$

where the convolution of two functions is defined by

$$(f*g)(t) = \int_0^t f(\xi)g(t-\xi)d\xi = \int_0^t f(t-\xi)g(\xi)d\xi.$$

Property 4: Some special natural transforms

$$\begin{aligned} \mathcal{N}^{+}[1] &= \frac{1}{s}, \\ \mathcal{N}^{+}[t] &= \frac{v}{s^2}, \\ \mathcal{N}^{+}\left[\frac{t^{n-1}}{(n-1)!}\right] &= \frac{v^{n-1}}{s^n}, n = 1,2,.... \end{aligned}$$

Property 5: If $\alpha > -1$, then the natural transform of t^{α} is given by

$$\mathcal{N}^{+}\left[t^{\alpha}\right] = \Gamma(\alpha+1)\frac{v^{\alpha}}{s^{\alpha+1}}.$$

Theorem 2.5. *If $R^{+}(s,v)$ is the natural transform of $f(t)$, then the natural transform of the Riemann-Liouville fractional integral for $f(t)$ of order α, is given by*

$$\mathcal{N}^{+}\left[I^{\alpha}f(t)\right] = \frac{v^{\alpha}}{s^{\alpha}}R^{+}(s,v). \tag{2.7}$$

Proof. The Riemann-Liouville fractional integral for the function $f(t)$, as in (2.1), can be expressed as the convolution

$$I^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)}t^{\alpha-1} * f(t). \tag{2.8}$$

Applying the natural transform in the Eq. (2.8) and using properties (3) and (5), we have

$$\begin{aligned} \mathcal{N}^{+}\left[I^{\alpha}f(t)\right] &= \mathcal{N}^{+}\left[\frac{1}{\Gamma(\alpha)}t^{\alpha-1} * f(t)\right] = v\frac{1}{\Gamma(\alpha)}\mathcal{N}^{+}\left[t^{\alpha-1}\right]\mathcal{N}^{+}\left[f(t)\right] \\ &= v\frac{v^{\alpha-1}}{s^{\alpha}}R^{+}(s,v) = \frac{v^{\alpha}}{s^{\alpha}}R^{+}(s,v). \end{aligned}$$

The proof is complete.

Theorem 2.6. *Let $n \in \mathbb{N}^*$ and $\alpha > 0$ be such that $n-1 < \alpha \leq n$ and $R^{+}(s,v)$ be the natural transform of the function $f(t)$, then the natural transform of the Caputo fractional derivative of the function $f(t)$ of order α, is given by*

$$\mathcal{N}^{+}\left[D^{\alpha}f(t)\right] = \frac{s^{\alpha}}{v^{\alpha}}R^{+}(s,v) - \sum_{k=0}^{n-1}\frac{s^{\alpha-(k+1)}}{v^{\alpha-k}}f^{(k)}(0). \tag{2.9}$$

Proof. Let $g(t) = f^{(n)}(t)$, then by the Definition 2.2 of the Caputo fractional derivative, we obtain

$$\begin{aligned} D^{\alpha} f(t) &= \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\xi)^{n-\alpha-1} f^{(n)}(\xi) d\xi \\ &= \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\xi)^{n-\alpha-1} g(\xi) d\xi \\ &= I^{n-\alpha} g(t). \end{aligned} \tag{2.10}$$

Applying the natural transform on both sides of (2.10) using Eq. (2.7), we get

$$\mathcal{N}^{+}\left[D^{\alpha} f(t)\right] = \mathcal{N}^{+}\left[I^{n-\alpha} g(t)\right] = \frac{v^{n-\alpha}}{s^{n-\alpha}} G^{+}(s,v). \tag{2.11}$$

Also, we have from the properties (1) and (2)

$$\mathcal{N}^{+}\left[g(t)\right] = \mathcal{N}^{+}\left[f^{(n)}(t)\right],$$

and

$$G^{+}(s,v) = \frac{s^n}{v^n} R^{+}(s,v) - \sum_{k=0}^{n-1} \frac{s^{n-(k+1)}}{v^{n-k}} \left[f^{(k)}(t)\right]_{t=0}.$$

Hence, (2.11) becomes

$$\begin{aligned} \mathcal{N}^{+}\left[D^{\alpha} f(t)\right] &= \frac{v^{n-\alpha}}{s^{n-\alpha}} \left(\frac{s^n}{v^n} R^{+}(s,v) - \sum_{k=0}^{n-1} \frac{s^{n-(k+1)}}{v^{n-k}} f^{(k)}(0) \right) \\ &= \frac{s^{\alpha}}{u^{\alpha}} R^{+}(s,v) - \sum_{k=0}^{n-1} \frac{s^{\alpha-(k+1)}}{v^{\alpha-k}} f^{(k)}(0). \end{aligned}$$

The proof is complete.

3 Reduced differential transform method

In this section, we present some important definitions and operations of the reduced differential transform method in which can help to better understand of the indicated method [11].
Now, assume that the function of two variables $u(x,t)$ will be described as a product of two different variable functions, i.e., $u(x,t) = \phi(x)\psi(t)$. The function $u(x,t)$ can be displayed due to the properties of the differential transform as follows:

$$u(x,t) = \left(\sum_{i=0}^{\infty} \Phi(i) x^i\right)\left(\sum_{j=0}^{\infty} \Psi(j) t^j\right) = \sum_{k=0}^{\infty} U_k(x) t^k,$$

where $U_k(x)$ is the converted function of the source function $u(x,t)$.

Definition 3.1. Let $u(x,t)$ is analytic and differentiated continuously function with regard to space x and time t, in the domain of interest, then the reduced differential transform of $u(x,t)$ is given by

$$U_k(x) = \sum_{k=0}^{\infty} \frac{1}{k!} \left[\frac{\partial^k}{\partial t^k} u(x,t)\right]_{t=t_0}, \tag{3.1}$$

Here the lowercase $u(x,t)$ represents the original function while the uppercase $U_k(x)$ stands for the reduced transformed function.

Definition 3.2. The reduced differential inverse transform of $U_k(x)$ is defined by

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x)(t-t_0)^k. \tag{3.2}$$

Combining equations (3.1) and (3.2), we have

$$u(x,t) = \sum_{k=0}^{\infty} \frac{1}{k!} \left[\frac{\partial^k}{\partial t^k} u(x,t) \right]_{t=t_0} (t-t_0)^k. \tag{3.3}$$

In particular, for $t_0 = 0$, equation (3.3) becomes

$$u(x,t) = \sum_{k=0}^{\infty} \frac{1}{k!} \left[\frac{\partial^k}{\partial t^k} u(x,t) \right]_{t=0} t^k. \tag{3.4}$$

From the above definitions, the fundamental operations of the reduced differential transform method are given by the following theorems.

Theorem 3.3. *Let $U_k(x), V_k(x)$ and $W_k(x)$ be the reduced differential transform of the functions $u(x,t), v(x,t)$ and $w(x,t)$ respectively, then*

(1) if

$$w(x,t) = \lambda u(x,t) + \mu v(x,t),$$

then

$$W_k(x) = \lambda U_k(x) + \mu V_k(x), \lambda, \mu \in \mathbb{R}.$$

(2) if

$$w(x,t) = u(x,t)v(x,t),$$

then

$$W_k(x) = \sum_{r=0}^{k} U_r(x) V_{k-r}(x).$$

(3) if

$$w(x,t) = u^1(x,t)u^2(x,t)...u^n(x,t),$$

then

$$W_k(x) = \sum_{k_{n-1}=0}^{k} \sum_{k_{n-2}=0}^{k_{n-1}} ... \sum_{k_2=0}^{k_3} \sum_{k_1=0}^{k_2} U^1_{k_1}(x) U^2_{k_2-k_1}(x) \times ... \times U^{n-1}_{k_{n-1}-k_{n-2}}(x) U^n_{k-k_{n-1}}(x).$$

(4) if

$$w(x,t) = \frac{\partial^n}{\partial t^n} u(x,t),$$

then

$$\begin{aligned} W_k(x) &= (k+1)(k+2)...(k+n)U_{k+n}(x) \\ &= \frac{(k+n)!}{k!} U_{k+n}(x), n = 1, 2, ... \end{aligned}$$

(5) if

$$w(x,t) = \frac{\partial^n}{\partial x^n} u(x,t),$$

then

$$W_k(x) = \frac{\partial^n}{\partial x^n} U_k(x), n = 1, 2, ...$$

4 Natural reduced differential transform method (NRDTM)

Theorem 4.1. *Suppose that we have a nonlinear time-fractional partial differential equation with initial conditions of the form*

$$D_t^\alpha u(x,t) = Lu(x,t) + Nu(x,t) + f(x,t), n-1 < \alpha \leq n, n \in \mathbb{N}^*, \tag{4.1}$$

$$\frac{\partial^k u(x,0)}{\partial t^k} = u^{(k)}(x,0) = g_k(x), k = 0,1,2,...,n-1, \tag{4.2}$$

where $D_t^\alpha u(x,t)$ is the Caputo fractional derivative of the function $u(x,t)$ of order α, L is a linear operator which has partial derivatives, N is a nonlinear operator and $f(x,t)$ is the source term. Then, by NRDTM, the approximate analytical solution of equations (4.1)-(4.2) is given in the form of infinite series which converges very rapidly to the exact solution as follows

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x),$$

where $U_k(x)$ is the reduced differential transformed function of $u(x,t)$.

Proof. In order to achieve our result, we consider the following nonlinear time-fractional partial differential equation (4.1) with the initial conditions (4.2).
Applying the natural transform on both sides of equation (4.1), we get

$$\mathcal{N}^+ \left[D_t^\alpha u(x,t)\right] = \mathcal{N}^+ \left[Lu(x,t) + Nu(x,t) + f(x,t)\right]. \tag{4.3}$$

Using the Theorem 2.5 and the initial conditions in equation (4.2), we obtain

$$\frac{s^\alpha}{v^\alpha}\mathcal{N}^+ \left[u(x,t)\right] - \sum_{k=0}^{n-1} \frac{s^{\alpha-(k+1)}}{v^{\alpha-k}} u^{(k)}(x,0) = \mathcal{N}^+ \left[Lu(x,t) + Nu(x,t)\right] + \mathcal{N}^+ \left[f(x,t)\right]. \tag{4.4}$$

Thus, we have

$$\mathcal{N}^+ \left[u(x,t)\right] = \sum_{k=0}^{n-1} \frac{v^k}{s^{k+1}} g_k(x) + \frac{v^\alpha}{s^\alpha}\mathcal{N}^+ \left[f(x,t)\right] + \frac{v^\alpha}{s^\alpha}\mathcal{N}^+ \left[Lu(x,t) + Nu(x,t)\right]. \tag{4.5}$$

Operating the inverse natural transform on both sides of equation (4.5), we get

$$u(x,t) = G(x,t) + \mathcal{N}^{-1}\left(\frac{v^\alpha}{s^\alpha}\mathcal{N}^+ \left[Lu(x,t) + Nu(x,t)\right]\right). \tag{4.6}$$

where $G(x,t)$, represents the term arising from the source term and the prescribed initial conditions.
Now, applying the reduced differential transform method definition, the following iteration formula can be defined as

$$\begin{aligned} U_0(x) &= G(x,t), \\ U_{k+1}(x) &= \mathcal{N}^{-1}\left(\frac{v^\alpha}{s^\alpha}\mathcal{N}^+ \left[LU_k(x) + NU_k(x)\right]\right), k \geq 0, \end{aligned} \tag{4.7}$$

where $LU_k(x)$ and $NU_k(x)$ are the reduced differential transform functions of $Lu(x,t)$ and $Nu(x,t)$, respectively.
From equation (4.7), we have

$$\begin{aligned} U_0(x) &= G(x,t), \\ U_1(x) &= \mathcal{N}^{-1}\left(\frac{v^\alpha}{s^\alpha}\mathcal{N}^+ \left[LU_0(x) + NU_0(x)\right]\right), \\ U_2(x) &= \mathcal{N}^{-1}\left(\frac{v^\alpha}{s^\alpha}\mathcal{N}^+ \left[LU_1(x) + NU_1(x)\right]\right), \\ U_3(x) &= \mathcal{N}^{-1}\left(\frac{v^\alpha}{s^\alpha}\mathcal{N}^+ \left[LU_2(x) + NU_2(x)\right]\right), \\ &\vdots \end{aligned}$$

Then, the approximate analytical solution of equations (4.1)-(4.2) is given as

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x).$$

The proof is complete.

5 Numerical Applications

In this section, some nonlinear time-fractional differential equations are considered to illustrate the accuracy and efficiency of the proposed method.

Example 5.1. Consider the nonlinear time-fractional gas dynamic equation subject to the initial condition of the form

$$D_t^{\alpha} u(x,t) + \frac{1}{2}\left(u^2(x,t)\right)_x = u(x,t) - u^2(x,t), 0 < \alpha \leq 1, \tag{5.1}$$

$$u(x,0) = e^{-x}, \tag{5.2}$$

where $D_t^{\alpha} u(x,t)$ is the Caputo time-fractional derivative of the function $u(x,t)$ of order α and $(x,t) \in \mathbb{R} \times \mathbb{R}^+$.
Applying the same methodology described in Section 4 to the equations (5.1)-(5.2), we have the following iteration formula

$$\begin{aligned} U_0(x) &= e^{-x}, \\ U_{k+1}(x) &= \mathcal{N}^{-1}\left(\frac{v^{\alpha}}{s^{\alpha}}\mathcal{N}^{+}\left(U_k(x) - A_k(x) - B_k(x)\right)\right), \end{aligned} \tag{5.3}$$

where $A_k(x)$ and $B_k(x)$ are the transformed values of the nonlinear terms, $\frac{1}{2}\left(u^2(x,t)\right)_x$ and $u^2(x,t)$, respectively . For the convenience of the reader, the first few nonlinear terms are as follows

$$\begin{aligned} A_0(x) &= \frac{1}{2}\left(U_0^2(x)\right)_x, \\ A_1(x) &= \frac{1}{2}\left(2U_0(x)U_1(x)\right)_x, \\ A_2(x) &= \frac{1}{2}\left(2U_0(x)U_2(x) + U_1^2(x)\right)_x. \end{aligned}$$

$$\begin{aligned} B_0(x) &= U_0^2(x), \\ B_1(x) &= 2U_0(x)U_1(x), \\ B_2(x) &= 2U_0(x)U_2(x) + U_1^2(x). \end{aligned}$$

By iterative calculation on relationship (5.3), we have

$$\begin{aligned} U_0(x) &= e^{-x}, \\ U_1(x) &= e^{-x}\frac{t^{\alpha}}{\Gamma(\alpha+1)}, \\ U_2(x) &= e^{-x}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \\ U_3(x) &= e^{-x}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}, \\ U_4(x) &= e^{-x}\frac{t^{4\alpha}}{\Gamma(4\alpha+1)}, \\ &\vdots \end{aligned}$$

Thus, the solution of equations (5.1)-(5.2) is given by

$$\begin{aligned} u(x,t) &= e^{-x}\left(1+\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\cdots\right) \\ &= e^{-x}\left(\sum_{k=0}^{\infty}\frac{\left(t^{k}\right)^{\alpha}}{\Gamma(k\alpha+1)}\right) \\ &= e^{-x}E_{\alpha}(t^{\alpha}), \end{aligned} \tag{5.4}$$

where $E_{\alpha}(t^{\alpha})$ denotes the Mittag-Leffler function defined by equation (2.4).
If we put $\alpha = 1$ in equation (5.4), we obtain the exact solution in closed form

$$u(x,t) = e^{t-x}.$$

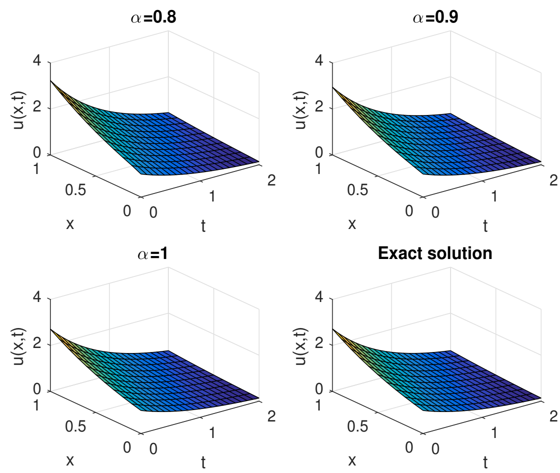

Figure 1. 3D plots graphs of the 4−term approximate solutions by NRDTM and exact solution for Example 5.1

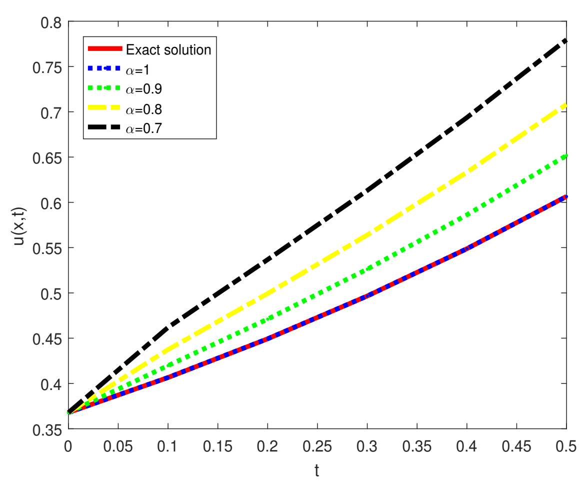

Figure 2. 2D plots graphs of the 4−term approximate solutions by NRDTM and exact solution for Example 5.1 when $x = 1$

t	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1$	exact solution	$\lvert u_{exact} - u_{NRDTM} \rvert$
0.01	0.38448	0.37797	0.37400	0.37158	0.37158	3.0708×10^{-13}
0.03	0.40495	0.39274	0.38458	0.37908	0.37908	7.4870×10^{-11}
0.05	0.42240	0.40606	0.39471	0.38674	0.38674	9.6606×10^{-10}
0.07	0.43864	0.41881	0.40471	0.39455	0.39455	5.2132×10^{-9}
0.09	0.45425	0.43128	0.41469	0.40252	0.40252	1.8377×10^{-8}

Table 1. The numerical values of the $4-$term approximate solutions by NRDTM and exact solution for Example 5.1 when $x = 1$

Example 5.2. Consider the nonlinear time-fractional reaction-diffusion-convection equation subject to the initial condition of the form

$$D_t^{\alpha} u(x,t) = u_{xx}(x,t) - u_x(x,t) + u(x,t)u_x(x,t) - u^2(x,t) + u(x,t), 0 < \alpha \leq 1, \quad (5.5)$$

$$u(x,0) = e^x, \quad (5.6)$$

where $D_t^{\alpha} u(x,t)$ is the Caputo time-fractional derivative of the function $u(x,t)$ of order α, and $(x,t) \in \mathbb{R} \times \mathbb{R}^+$.
Applying the same methodology described in Section 4 to the equations (5.5)-(5.6), we have the following iteration formula

$$\begin{aligned} U_0(x) &= e^x, \\ U_{k+1}(x) &= \mathcal{N}^{-1}\left(\frac{v^{\alpha}}{s^{\alpha}} \mathcal{N}^+ \left(U_{kxx}(x) - U_{kx}(x) + A_k(x) - B_k(x) + U_k(x)\right)\right), \end{aligned} \quad (5.7)$$

where $A_k(x)$ and $B_k(x)$ are the transformed values of the nonlinear terms, $u(x,t)u_x(x,t)$ and $u^2(x,t)$, respectively . For the convenience of the reader, the first few nonlinear terms are as follows

$$\begin{aligned} A_0(x) &= U_0(x)U_{0x}(x), \\ A_1(x) &= U_0(x)U_{1x}(x) + U_1(x)U_{0x}(x), \\ A_2(x) &= U_0(x)U_{2x}(x) + U_1(x)U_{1x}(x) + U_2(x)U_{0x}(x). \end{aligned}$$

$$\begin{aligned} B_0(x) &= U_0^2(x), \\ B_1(x) &= 2U_0(x)U_1(x), \\ B_2(x) &= 2U_0(x)U_2(x) + U_1^2(x). \end{aligned}$$

By iterative calculation on relationship (5.7), we have

$$\begin{aligned} U_0(x) &= e^x, \\ U_1(x) &= e^x \frac{t^{\alpha}}{\Gamma(\alpha+1)}, \\ U_2(x) &= e^x \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \\ U_3(x) &= e^x \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}, \\ U_4(x) &= e^x \frac{t^{4\alpha}}{\Gamma(4\alpha+1)}, \\ &\vdots \end{aligned}$$

Thus, the solution of equations (5.5)-(5.6) is given by

$$\begin{aligned} u(x,t) &= e^{x}\left(1+\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+...\right) \\ &= e^{x}\left(\sum_{k=0}^{\infty}\frac{(t^{\alpha})^{k}}{\Gamma(k\alpha+1)}\right) \\ &= e^{x}E_{\alpha}(t^{\alpha}), \end{aligned} \tag{5.8}$$

where $E_{\alpha}(t^{\alpha})$ denotes the Mittag-Leffler function defined by equation (2.4).
If we put $\alpha = 1$ in equation (5.8), we obtain the exact solution in closed form

$$u(x,t) = e^{t+x}.$$

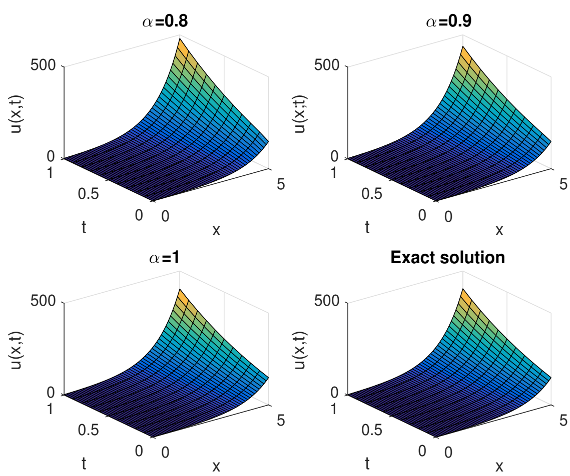

Figure 3. 3D plots graphs of the 4−term approximate solutions by NRDTM and exact solution for Example 5.2

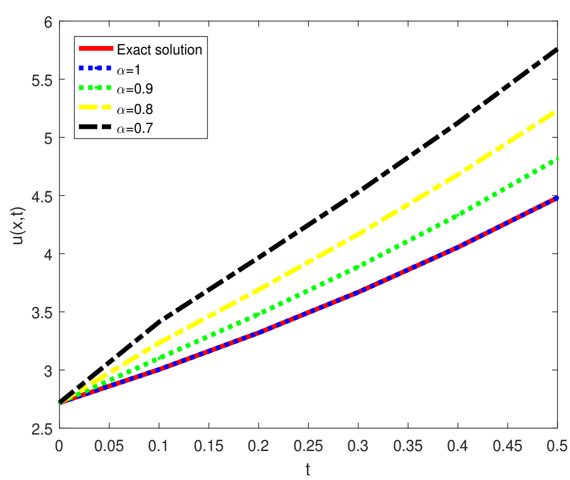

Figure 4. 2D plots graphs of the 4−term approximate solutions by NRDTM and exact solution for Example 5.2 when $x = 1$

t	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1$	exact solution	$\lvert u_{exact} - u_{NRDTM}\rvert$
0.01	2.8409	2.7928	2.7635	2.7456	2.7456	2.2690×10^{-12}
0.03	2.9922	2.9020	2.8417	2.8011	2.8011	5.5322×10^{-10}
0.05	3.1212	3.0004	2.9165	2.8577	2.8577	7.1383×10^{-9}
0.07	3.2412	3.0946	2.9904	2.9154	2.9154	3.8520×10^{-8}
0.09	3.3565	3.1868	3.0642	2.9743	2.9743	1.3579×10^{-7}

Table 2. The numerical values of the 4−term approximate solutions by NRDTM and exact solution for Example 5.2 when $x = 1$

Example 5.3. Consider the nonlinear time-fractional wave-like equation with variable coefficients subject to the initial conditions of the form

$$D_t^\alpha u(x,t) = x^2 \frac{\partial}{\partial x}(u_x(x,t)u_{xx}(x,t)) - x^2(u_{xx}^2(x,t)) - u(x,t), 1 < \alpha \leq 2, \tag{5.9}$$

$$u(x,0) = 0, u_t(x,0) = x^2, \tag{5.10}$$

where $D_t^\alpha u(x,t)$ is the Caputo time-fractional derivative of the function $u(x,t)$ of order α, and $(x,t) \in \,]0,1[\, \times \mathbb{R}^+$.
Applying the same methodology described in Section 4 to the equations (5.9)-(5.10), we have the following iteration formula

$$\begin{aligned} U_0(x) &= tx^2, \\ U_{k+1}(x) &= \mathcal{N}^{-1}\left(\frac{v^\alpha}{s^\alpha}\mathcal{N}^+\left(x^2\frac{\partial}{\partial x}A_k(x) - x^2B_k(x) - U_k(x)\right)\right), \end{aligned} \tag{5.11}$$

where $A_k(x)$ and $B_k(x)$ are the transformed values of the nonlinear terms, $u_x(x,t)u_{xx}(x,t)$ and $u_{xx}^2(x,t)$, respectively . For the convenience of the reader, the first few nonlinear terms are as follows

$$\begin{aligned} A_0(x) &= U_{0x}(x)U_{0xx}(x), \\ A_1(x) &= U_{0x}(x)U_{1xx}(x) + U_{1x}(x)U_{0xx}(x), \\ A_2(x) &= U_{0x}(x)U_{2xx}(x) + U_{1x}(x)U_{1xx}(x) + U_{2x}(x)U_{0xx}(x). \end{aligned}$$

$$\begin{aligned} B_0(x) &= U_{0xx}^2(x), \\ B_1(x) &= 2U_{0xx}(x)U_{1xx}(x), \\ B_2(x) &= 2U_{0xx}(x)U_{2xx}(x) + U_{1xx}^2(x). \end{aligned}$$

By iterative calculation on relationship (5.11), we have

$$\begin{aligned} U_0(x) &= tx^2, \\ U_1(x) &= -\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}x^2, \\ U_2(x) &= \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}x^2, \\ U_3(x) &= -\frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)}x^2, \\ U_4(x) &= \frac{t^{4\alpha+1}}{\Gamma(4\alpha+2)}x^2, \\ &\vdots \end{aligned}$$

Thus, the solution of equations (5.9)-(5.10) is given by

$$\begin{aligned} u(x,t) &= x^2\left(t-\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}-\frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)}+...\right) \\ &= x^2\left(t\sum_{k=0}^{\infty}\frac{(-t^{\alpha})^k}{\Gamma(k\alpha+2)}\right) \\ &= x^2\left(tE_{\alpha,2}(-t^{\alpha})\right), \end{aligned} \tag{5.12}$$

where $E_{\alpha,2}(-t^{\alpha})$ denotes the Mittag-Leffler function defined by equation (2.5).
If we put $\alpha=2$ in equation (5.12), we obtain the exact solution in closed form

$$u(x,t)=x^2\sin(t).$$

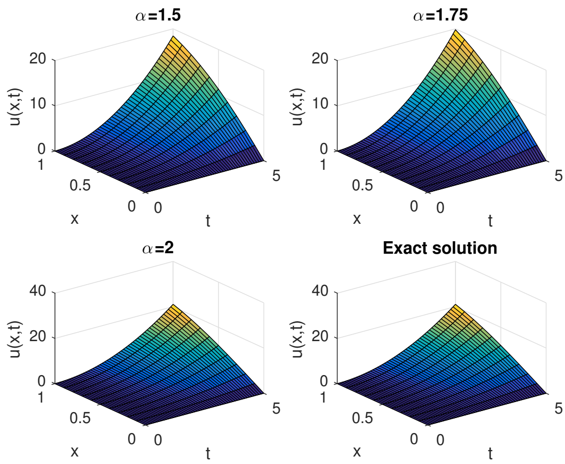

Figure 5. 3D plots graphs of the 4−term approximate solutions by NRDTM and exact solution for Example 5.3

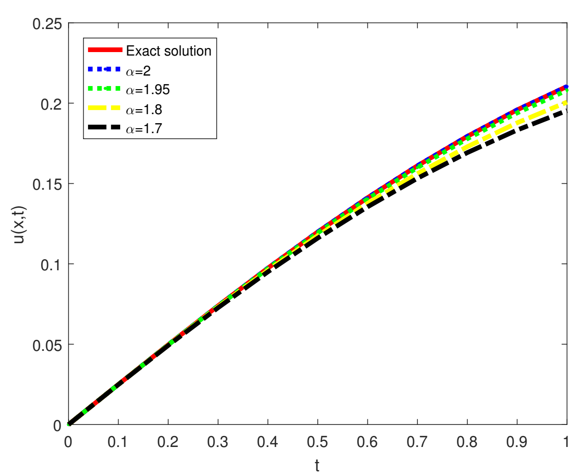

Figure 6. 2D plots graphs of the 4−term approximate solutions by NRDTM and exact solution for Example 5.3 when $x=0.5$

t	$\alpha = 1.7$	$\alpha = 1.8$	$\alpha = 1.95$	$\alpha = 2$	exact solution	$\lvert u_{exact} - u_{NRDTM} \rvert$
0.1	0.02488	0.02492	0.02495	0.02496	0.02496	6.8887×10^{-16}
0.3	0.07271	0.07319	0.07374	0.07388	0.07388	1.3549×10^{-11}
0.5	0.11604	0.11752	0.11934	0.11986	0.11986	1.3425×10^{-9}
0.7	0.15325	0.15615	0.15994	0.16105	0.16105	2.7677×10^{-8}
0.9	0.18327	0.18777	0.19394	0.19583	0.19583	2.6495×10^{-7}

Table 3. The numerical values of the $4-$term approximate solutions by NRDTM and exact solution for Example 5.3 when $x = 0.5$

6 Conclusion

In this paper, we have presented a combination of the natural transform method and the reduced differential transform method to obtain the exact solution of nonlinear time-fractional partial differential equations. This combination creates a strong method called the natural reduced differential transfrm method (NRDTM). This method has been successfully applied to three different numerical applications. the NRDTM is an analytical method and runs by using the initial conditions only. Thus, it can be used to solve equations with fractional and integer order with respect to time. An important advantage of the new approach is its low computational load. Our goal in the future is to apply the NRDTM to other nonlinear fractional partial differential equations that arise in other fields of science, but with different fractional derivative operators such as: Caputo-Fabrizio fractional derivative, Atangana-Baleanu-Caputo fractional derivative, Conformable fractional derivative, etc.

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