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ON THE LIMIT OF A DIFFERENCE EQUATION WITH A GENERATING SEQUENCE

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Abstract In this paper we study systems of difference equations numerically and theoretically. These systems were considered by many researchers. We will focus on the general form of the solution and the limits. We use in certain cases the computer to verify the limit properties. In all the systems the first equation is independent of the second equation.

1 Introduction

Difference equations appear as natural descriptions of observed evolution phenomena because measurements of time evolving variables are discrete and as such, these equations are in their own right important mathematical models. More importunately, difference equations also appear in the study of discrimination methods for difference equations. Several results in the theory of difference equation have been obtained as more or less natural discrete analogues of corresponding results of difference equation. Recently many researchers worked in the topic of the behavior of the solution of difference equations. Stevic, Kurbanli and Elsayed are working recently on this topic cf. [8], [4] and cf. [3], especially on the rational difference equations.

In [1] the authors studied the global stability, the boundedness character and the periodic nature of the positive solutions of the recursive sequence

$$x_{n+1} = \alpha + \frac{x_{n+1}}{x_n}$$

where the initial conditions x_{-1}, x_0 and α are arbitrary positive real numbers. In [9] the authors established that every positive solution of the difference equation

$$x_{n+1} = \frac{A}{x_n} + \frac{1}{x_{n-2}},$$

where A is a positive constant, converges to a periodic two solution. In [8] Stevic proved that if $A > 1$ then every positive solution of

$$x_{n+1} = \frac{A}{\prod_{i=0}^{k} x_{n-i}} + \frac{1}{\prod_{j=k+2}^{2(k+1)} x_{n-j}}$$

converges to a period $k + 2$ solution. The following difference equation

$$x_{n+1} = max\left\{\frac{1}{x_n}, \frac{A_n}{x_{n-1}}\right\},$$

was studied by El-Metwally, Elabbasy and Elsayed (cf. [6]). In some cases they found the general form of the solution, also they proved that every positive solution of this equation is bounded. In [3] Elsayed computed the general form of the solutions of difference equation

$$x_{n+1} = \frac{x_{n-5}}{1 + x_{n-1}\, x_{n-3} x_{n-5}}.$$

Further, he proved that every positive solution of this equation is bounded and

$$\lim_{n\to\infty} x_n = 0.$$

In [4] Kurbanli consider the following system

$$x_{\mathrm{n}+1} = \frac{x_{\mathrm{n}-1}}{x_{\mathrm{n}-1}y_n - 1}, \; y_{\mathrm{n}+1} = \frac{y_{\mathrm{n}-1}}{x_n y_{\mathrm{n}-1} - 1}, \text{ and } z_{\mathrm{n}+1} = \frac{z_{\mathrm{n}-1}}{y_n z_{\mathrm{n}-1} - 1}.$$

He derived a formula for x_n , y_n , z_n. In [5] the following system of equations was studied by Ibrahim

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} \frac{x_{n-1}}{x_{n-1}+r} \\ \frac{y_{n-1}}{x_n y_{n-1}+r} \end{pmatrix}$$

where r is a fixed real number and the following initial condition

$$x_0 = b, x_{-1} = c, y_0 = a, \; y_{-1} = d.$$

In[5] Ibrahim proved the following result: Let a ,b ,c ,d and r be positive real numbers. Then, the general solution of the system is

$$x_{2k} = \frac{b}{G(b,k)}, x_{2k+1} = \frac{c}{G(c,k+1)},$$

$$y_{2k} = \frac{ac^k}{ac^k + a\sum_{i=2}^{k} c^{k-i+1} \, r^{i-1} \prod_{j=0}^{i-2} G(c,k-j) + r^k \prod_{j=0}^{k-1} G(c,k-j)},$$

$$y_{2k+1} = \frac{db^{k+1}}{db^{k+1} + db\sum_{i=2}^{k} b^{k-i+1} r^{i-1} \prod_{j=0}^{i-2} G(b,k-j) + r^k \prod_{j=0}^{k-1} G(b,k-j)}.$$

where

$$G(c,0) = c + r, G(c,i) = c + rG(c,i-1).$$

In [2] Bany Khaled considered the system

$$x_{n+1} = \frac{x_{n-1}}{x_n + r}, y_{n+1} = \frac{x_{n-1}y_{n-1}}{x_{n-1}y_{n-1} + r}$$

with initial values

$$x_{-1} = a, x_0 = 0, y_{-1} = b.$$

Hence, according to definition we obtain

$$x_{2k} = 0, y_{2k} = 0.$$

Bany Khaled proved an estimate for the solution. Based on it she proved: If $a, b > 0$ and $r > 1$ such that $a^2 < r$, then

$$\lim_{k\to\infty} x_{2k+1} = 0, \; \lim_{k\to\infty} y_{2k+1} = 0.$$

2 Two Simple Systems

We are interested in a sequence, where the coefficients are generated by a sequence or a generating function in general. Hence, it is easier to write it in the form of a system. In this paper we consider the following systems

$$x_{n+1} = \frac{x_n}{x_n + 1}, \quad y_{n+1} = \frac{x_{n-1}y_n}{x_{n-1}y_n + 1} \tag{2.1}$$

$$x_{n+1} = \frac{x_{n-1}}{x_{n-1} + 1}, \quad y_{n+1} = \frac{x_{n-1}y_{n-1}}{x_{n-1}y_{n-1} + 1} \tag{2.2}$$

We define

$$W(p,f) = \sum_{k=0}^{f} \frac{1}{\Gamma(k+p)}, R(b) = \Gamma(b) - \Gamma(b,1). \tag{2.3}$$

We verified the following result by Mathematica for $p > 0$:

$$\sum_{j=0}^{f} \frac{1}{\Gamma(j+p)} = e\frac{(p-1)\,R\,(p-1)}{\Gamma(p)} - e\frac{(f+p)\,R\,(f+p)}{\Gamma(f+1+p)} \tag{2.4}$$

where e is the Euler number (approx. 2.718) and $\Gamma(a,x)$ is the incomplete gamma function. We verified also the following summation laws by Mathematica for $m, n > 0$:

$$\prod_{l=1}^{m-1} \frac{l}{l+d} = \frac{\Gamma(d{+}1)\,(m-1)!}{\Gamma(d+m)} \tag{2.5}$$

$$\sum_{l=1}^{n-1} \frac{\Gamma(d{+}l)}{(l-1)!} = \frac{\Gamma(d{+}n)}{(d+1)\,(n-2)\,!} \tag{2.6}$$

Hence, we obtain

$$\prod_{l=1}^{m} \frac{l}{l+d} = \frac{\Gamma(d{+}1)m!}{\Gamma(d+m+1)}, \sum_{m=2}^{n-1} \frac{\Gamma(d{+}m)}{(m-1)!} = \frac{\Gamma(d{+}n)}{(d+1)\,(n-2)\,!} - \Gamma\,(d+1) \tag{2.7}$$

$$\sum_{m=3}^{n} \frac{\Gamma(d+m)}{(m-1)!} = \frac{\Gamma(d+n+1)}{(d+1)\,(n-1)\,!} - \Gamma\,(d+1) - \frac{\Gamma\,(d{+}2)}{(2-1)!} =$$

$$\frac{\Gamma(d{+}n+1)}{(d+1)\,(n-1)\,!} - \Gamma\,(d+1) - \Gamma(d{+}2) \tag{2.8}$$

$$\sum_{m=2}^{n} \frac{\Gamma(d{+}m+1)}{m!} = \sum_{j=3}^{n+1} \frac{\Gamma(d{+}j)}{(j-1)!} = \frac{\Gamma(d{+}n+2)}{(d+1)\,n!} - \Gamma\,(d+1) - \Gamma\,(d{+}2) \tag{2.9}$$

2.1 The general solution of system (2.1)

We study now the system (2.1) with initial values

$$x_0 = a, x_{-1} = b, y_0 = c.$$

We find that in general

$$x_n = \frac{a}{na+1}, y_n = \frac{a^{n-1}bc}{P_n}, \text{for } n = 1, 2, \ldots$$

$$P_{n+1} = \; a^n bc + ((n-1)\,a+1)\,P_n \;\;, \quad P_1 = bc{+}1.$$

Hence

$$P_n = a^{n-1} bc\Gamma\left(n - 2 + \frac{1+a}{a}\right) * W\,(a, n-2) + Z,$$

where

$$Z = \frac{a^{n-2}\Gamma\left(n-2+\frac{1+a}{a}\right)}{\Gamma\left(\frac{1+a}{a}\right)} P_1.$$

We reach the following result

Proposition 2.1. *The general solution of the system (2.1) is*

$$x_1 = \frac{a}{a+1}, y_1 = \frac{bc}{bc+1}, x_n = \frac{a}{na+1},$$

$$y_n = \frac{abce^{-1}\Gamma(1+a^{-1})\Gamma(n+a^{-1})}{(n+\frac{1}{a}-1)(\Gamma(n+\frac{1}{a}-1)R(\frac{1}{a}) - \Gamma(\frac{1}{a})R(n+a^{-1}-1)) + \Gamma(n+\frac{1}{a}-1)}, n > 1.$$

Proof. We concluded previously that

$$y_n = \frac{\Gamma\left(1+a^{-1}\right)}{\Gamma\left(n-1+a^{-1}\right)} \frac{a^{n-1}bc}{a^{n-1}bc\Gamma\left(1+a^{-1}\right)W\left(a,n-2\right)+a^{n-2}(bc+1)}$$

If we set $p = 1 + a^{-1}$ in (2.4), then we obtain for n=2, 3, 4, ...

$$W\left(a,n-2\right) = \frac{a^{-1}e(\Gamma\left(a^{-1}\right)-\Gamma\left(a^{-1},1\right))}{\Gamma\left(1+a^{-1}\right)} -$$

$$\frac{e\left(n-1+a^{-1}\right)\left(\Gamma\left(n-1+a^{-1}\right)-\Gamma\left(n-1+a^{-1},1\right)\right)}{\Gamma\left(n+a^{-1}\right)}, y_n = \frac{\Gamma\left(1+a^{-1}\right)}{\Gamma\left(n-1+a^{-1}\right)}$$

$$\frac{abc}{abc\Gamma\left(1+a^{-1}\right)W\left(a,n-2\right)+bc+1} = \frac{abce^{-1}\Gamma\left(1+a^{-1}\right)\Gamma\left(n+a^{-1}\right)}{H},$$

where H denotes the quantity

$$\Gamma\left(n-1+a^{-1}\right)+bc(\Gamma\left(n+a^{-1}\right)R\left(a^{-1}\right)+\Gamma\left(n-1+a^{-1}\right)-$$

$$\left(an+1-a\right)\Gamma\left(1+a^{-1}\right)R\left(n+a^{-1}-1\right)) =$$

$$\Gamma\left(n+a^{-1}-1\right)a^{-1}\left(\left(an+1-a\right)R\left(a^{-1}\right)+a\right)-\left(an+1-a\right)\Gamma\left(1+a^{-1}\right)R\left(n+a^{-1}-1\right) =$$

$$\left(n+a^{-1}-1\right)\left(\Gamma\left(n+a^{-1}-1\right)R\left(a^{-1}\right)-\ \Gamma\left(a^{-1}\right)R\left(n+a^{-1}-1\right)\right)+\Gamma\left(n+a^{-1}-1\right),$$

since

$$\Gamma\left(n+a^{-1}\right)R\left(a^{-1}\right)+\Gamma\left(n-1+a^{-1}\right) = \Gamma\left(n+a^{-1}-1\right)\left(\left(n-1+a^{-1}\right)R\left(a^{-1}\right)+1\right) =$$

$$\Gamma\left(n+a^{-1}-1\right)a^{-1}\left(\left(an+1-a\right)R\left(a^{-1}\right)+a\right).$$

The proof is complete. □

Remark 2.2. We consider for example: $a = b = c = 3$. Then the solution of the system (2.1) is according to the formula

$$W(0) = \sum_{i=0}^{0} \frac{1}{\Gamma\left(0+2-i+\frac{1}{3}\right)} = \frac{9}{8}\frac{\sqrt{3}}{\pi}\Gamma\left(\frac{2}{3}\right),$$

$$y_3 = \frac{9\Gamma(\frac{1}{3})}{\Gamma(3-1+\frac{1}{3})(9\Gamma\left(\frac{1}{3}\right)*\frac{9}{8}\frac{\sqrt{3}}{\pi}\Gamma\left(\frac{2}{3}\right)+3*3*3+3*3+1)} = \frac{81}{229},$$

$$W(1) = \sum_{i=0}^{1} \frac{1}{\Gamma\left(1+2-i+\frac{1}{3}\right)} = \frac{45}{28}\frac{\sqrt{3}}{\pi}\Gamma\left(\frac{2}{3}\right),$$

$$y_4 = \frac{9\Gamma(\frac{1}{3})}{\Gamma(4-1+\frac{1}{3})(9\Gamma\left(\frac{1}{3}\right)*\frac{45}{28}\frac{\sqrt{3}}{\pi}\Gamma\left(\frac{2}{3}\right)+27+9+1)} = \frac{243}{1846},$$

$$W(2) = \sum_{i=0}^{2} \frac{1}{\Gamma\left(2+2-i+\frac{1}{3}\right)} = \frac{981}{560}\frac{\sqrt{3}}{\pi}\Gamma\left(\frac{2}{3}\right),$$

$$y_5 = \frac{9\Gamma(\frac{1}{3})}{\Gamma(5-1+\frac{1}{3})(9\Gamma\left(\frac{1}{3}\right)*\frac{981}{560}\frac{\sqrt{3}}{\pi}\Gamma\left(\frac{2}{3}\right)+27+9+1)} = \frac{729}{19189}.$$

On the other hand

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} \frac{x_1}{x_1+1} \\ \frac{x_0y_1}{x_0y_1+1} \end{pmatrix} = \begin{pmatrix} \frac{3}{7} \\ \frac{27}{37} \end{pmatrix}, \begin{pmatrix} x_3 \\ y_3 \end{pmatrix} = \begin{pmatrix} \frac{x_2}{x_2+1} \\ \frac{x_1y_2}{x_1y_2+1} \end{pmatrix} = \begin{pmatrix} \frac{\frac{3}{7}}{\frac{3}{7}+1} \\ \frac{\frac{3}{4}*\frac{27}{37}}{\frac{3}{4}*\frac{27}{37}+1} \end{pmatrix} = \begin{pmatrix} \frac{3}{10} \\ \frac{81}{229} \end{pmatrix},$$

$$\begin{pmatrix} x_4 \\ y_4 \end{pmatrix} = \begin{pmatrix} \frac{x_3}{x_3+1} \\ \frac{x_2y_3}{x_2y_3+1} \end{pmatrix} = \begin{pmatrix} \frac{\frac{3}{10}}{\frac{3}{10}+1} \\ \frac{\frac{3}{7}*\frac{81}{229}}{\frac{3}{7}*\frac{81}{229}+1} \end{pmatrix} = \begin{pmatrix} \frac{3}{13} \\ \frac{243}{1846} \end{pmatrix},$$

$$\begin{pmatrix} x_5 \\ y_5 \end{pmatrix} = \begin{pmatrix} \frac{x_4}{x_4+1} \\ \frac{x_3y_4}{x_3y_4+1} \end{pmatrix} = \begin{pmatrix} \frac{\frac{3}{13}}{\frac{3}{13}+1} \\ \frac{\frac{3}{10}*\frac{243}{1846}}{\frac{3}{10}*\frac{243}{1846}+1} \end{pmatrix} = \begin{pmatrix} \frac{3}{16} \\ \frac{729}{19189} \end{pmatrix}.$$

2.2 The general solution of system (2.2)

We consider now the system (2.2) with the following general initial values

$$x_{-1} = a, x_0 = c, y_{-1} = b, y_0 = d.$$

Proposition 2.3. *If $a, c > 0$, then the general solution of the system (2.2) is*

$$x_{2k} = \frac{c}{ck+1}, x_{2k+1} = \frac{a}{a(k+1)+1},$$

$$y_{2k} = \frac{\Gamma(\frac{1}{c})}{\Gamma(\frac{1}{c}) + \Gamma(\frac{1}{c})\Gamma(k+\frac{1}{c})W(\frac{1}{c}+2, k-3) + \Gamma(k+\frac{1}{c})(c+\frac{1}{d})},$$

$$y_{2k+1} = \frac{\Gamma(\frac{1}{a})}{\Gamma(\frac{1}{a}) + \Gamma(\frac{1}{a})\Gamma(k+\frac{1+a}{a})W(\frac{1}{a}+2, k-2) + \Gamma(k+\frac{1+a}{a})(a+\frac{1}{b})} \text{ for } k = 3, 4, 5, \ldots$$

Proof. According to definition

$$x_1 = \frac{x_{-1}}{x_{-1}+1} = \frac{a}{a+1} = \frac{a}{G(1)}, y_1 = \frac{x_{-1}y_{-1}}{x_{-1}y_{-1}+1} = \frac{ab}{ab+1} = \frac{ab}{H(1)},$$

where we denote by $G(n)$ (res. $H(n)$) the denominator of x_n (res. y_n). Since the variables x_n and y_n are separated in the even and the odd cases we are going to consider just one case. Now, we obtain

$$x_2 = \frac{x_0}{x_0+1} = \frac{c}{c+1} = \frac{c}{G(2)}, x_3 = \frac{x_1}{x_1+1} = \frac{\frac{a}{G(1)}}{\frac{a}{G(1)}+1} = \frac{a}{a+G(1)} = \frac{a}{G(3)}, \ldots,$$

$$y_7 = \frac{x_5y_5}{x_5y_5+1} = \frac{\frac{a}{G(5)} * \frac{a^3b}{H(5)}}{\frac{a}{G(5)} * \frac{a^3b}{H(5)}+1} = \frac{a^4b}{a^4b+G(5)H(5)} = \frac{a^4b}{H(7)}.$$

In general we denote by

$$G_j(a) = aj+1.$$

We conclude that

$$x_{2k} = \frac{c}{G_k(c)}, x_{2k+1} = \frac{a}{G_{k+1}(a)}, y_{2k+1} = \frac{a^{k+1}b}{H(2k+1)}, y_{2k} = \frac{c^kd}{H(2k)},$$

$$\begin{aligned} H(1) &= ab+1, H(3) = a^2b+G(1)H(1) = a^2b+G_1(a)(ab+1), H(5) = a^3b+G(3)H(3) = \\ &a^3b+G_2(a)(a^2b+G_1(a)(ab+1)) = a^3b+a^2bG_2(a)+G_1(a)G_2(a)(ab+1), \\ H(7) &= a^4b+G(5)H(5) = a^4b+G_3(a)(a^3b+a^2bG_2(a)+G_1(a)G_2(a)(ab+1)) \\ &= a^4b+a^3bG_3(a)+a^2bG_2(a)G_3(a)+G_1(a)G_2(a)G_3(a)(ab+1). \end{aligned}$$

We use the notation

$$B_n(a) = \prod_{j=1}^{n} G_j(a).$$

We rewrite

$$H(2*3+1) = a^4 b + a^3 b \frac{B_3(a)}{B_2(a)} + a^2 b \frac{B_3(a)}{B_1(a)} + B_3(a)(ab+1).$$

Thus the general form for $k = 3,\ 4,\ 5, ...$

$$H(2k+1) = a^{k+1} b + \sum_{i=1}^{k-1} a^{k+1-i} b \frac{B_k(a)}{B_{k-i}(a)} + B_k(a)(ab+1),$$

$$\sum_{i=1}^{k-1} a^{k+1-i} b \frac{B_k(a)}{B_{k-i}(a)} = a^{k+1} b B_k(a) \sum_{i=1}^{k-1} \frac{a^{-i}}{B_{k-i}(a)},$$

since

$$B_n(a) = \prod_{j=1}^{n} G_j(a) = \prod_{j=1}^{n} (aj+1).$$

But

$$\prod_{l=0}^{n} (p+ql) = q^{n+1} \Gamma\left(n + \frac{q+p}{q}\right) \Gamma^{-1}\left(\frac{p}{q}\right).$$

Hence,

$$B_n(a) = a^{n+1} \Gamma\left(n + \frac{1+a}{a}\right) \Gamma^{-1}\left(\frac{1}{a}\right),$$

$$a^{k+1} b B_k(a) \sum_{i=1}^{k-1} \frac{a^{-i} r^i}{B_{k-i}(a)} = a^{k+1} b a^{k+1} \frac{\Gamma\left(k + \frac{1+a}{a}\right)}{\Gamma\left(\frac{1}{a}\right)} \sum_{i=1}^{k-1} \frac{a^{-i} \Gamma\left(\frac{1}{a}\right)}{a^{k-i+1} \Gamma\left(k - i + \frac{1+a}{a}\right)} =$$

$$b a^{k+1} \Gamma\left(k + \frac{1+a}{a}\right) \sum_{i=1}^{k-1} \frac{1}{\Gamma(k - i + \frac{1+a}{a})} = b a^{k+1} \Gamma\left(k + \frac{1+a}{a}\right) \sum_{i=2}^{k} \frac{1}{\Gamma(i + \frac{1}{a})},$$

$$\frac{H(2k+1)}{a^{k+1}} = b\left(1 + \Gamma(k + \frac{1+a}{a})\right) W\left(\frac{1}{a} + 2, k-2\right) + \Gamma\left(k + \frac{1+a}{a}\right) \Gamma\left(\frac{1}{a}\right)^{-1} (a + b^{-1}),$$

$$y_{2k+1} = \frac{b \Gamma\left(a^{-1}\right)}{\Gamma\left(a^{-1}\right)\left(1 + \Gamma\left(k+1+a^{-1}\right)\right) W\left(2 + a^{-1}, k-2\right) + \Gamma\left(k+1+a^{-1}\right)\left(a + b^{-1}\right)}.$$

Similarly we can prove the other case. □

Corollary 2.4. *If $a, b, c, d > 0$, then the solution of the system (2.2) converges to zero.*

Proof. Since

$$W\left(2 + a^{-1}, k-2\right) = \sum_{l=0}^{k-2} \frac{1}{\Gamma\left(l + 2 + a^{-1}\right)},$$

then it is an increasing function in k. So the denominator of y_{2k+1} increases to infinity. Similarly we can prove that y_{2k+1}, x_{2k+1} and x_{2k} converge to zero other case. □

3 Generalization of System (2.1)

We generalize system (2.1) by replacing the form of x_n by a more general form as follows

$$x_n = \frac{a}{G(n)}, y_{n+1} = \frac{x_{n-1}y_n}{x_{n-1}y_n + 1}, n = 0, 1, 2, \ldots$$

where $y_0 = c$, $G(n), n = -1, 0, 1, \ldots$ are known nonzero values,

$$y_1 = \frac{x_{-1}y_0}{x_{-1}y_0 + 1} = \frac{\frac{a}{G(-1)}c}{\frac{a}{G(-1)}c + 1} = \frac{ac}{ac + G(-1)} = \frac{ac}{P(1)},$$

$$y_2 = \frac{x_0 y_1}{x_0 y_1 + 1} = \frac{\frac{a}{G(0)}\frac{ac}{P(1)}}{\frac{a}{G(0)}\frac{ac}{P(1)} + 1} = \frac{a^2 c}{a^2 c + G(0)P(1)} = \frac{a^2 c}{P(2)},$$

$$y_3 = \frac{x_1 y_2}{x_1 y_2 + 1} = \frac{\frac{a}{G(1)}\frac{a^2 c}{P(2)}}{\frac{a}{G(1)}\frac{a^2 c}{P(2)} + 1} = \frac{a^3 c}{a^3 c + G(1)P(2)} = \frac{a^3 c}{P(3)}.$$

We obtain in general

$$y_n = \frac{a^n c}{P(n)} \text{ for } n = 1, 2, 3, \ldots$$

where

$$P(n) = a^n c + G(n-2)P(n-1) \text{ for } n = 1, 2, 3, \ldots$$

We make the convention $P(0)=1$. In details:

$$P(2) = a^2 c + G(0)P(1) = a^2 c + acG(0) + G(-1)G(0),$$
$$P(3) = a^3 c + G(1)P(2) = a^3 c + G(1)(a^2 c + acG(0) + G(-1)G(0)) =$$
$$a^3 c + a^2 cG(1) + acG(0)G(1) + G(-1)G(0)G(1).$$

In general when $n = 4, 5, 6, \ldots$

$$P(n) = a^n c + a^{n-1}cG(n-2) + \cdots + acG(0)G(1)\ldots G(n-2) + G(-1)G(0)\ldots G(n-2) =$$
$$c(a^n + a^{n-1}G(n-2) + \cdots + aG(0)G(1)\ldots G(n-2)) + G(-1)G(0)G(1)\ldots G(n-2) =$$
$$cB(-1, n-2)\sum_{m=1}^{n} \frac{a^m}{B(-1, m-2)} + B(-1, n-2) =$$
$$B(-1, n-2)\left(c\sum_{m=1}^{n} \frac{a^m}{B(-1, m-2)} + 1\right).$$

Now, we are going to consider two cases regarding the form of $G(l)$. We study the limit of the solution in both cases and prove that

$$\lim_{n\to\infty} y_n = 0.$$

Case 1: If we choose

$$G(l) = \frac{l+2}{l+2+d}, d \neq -1, -2, \ldots$$

then we obtain $G(l) \neq 0$ for all $l = -1, 0, 1, \ldots$. Also,

$$B(-1, m-2) = \prod_{l=1}^{m} \frac{l}{l+d} = \frac{\Gamma(d+1)\Gamma(m+1)}{\Gamma(d+m+1)} = \cdots =$$

$$\frac{\Gamma(d+1)m!}{(d+m)\ldots(d+1)\Gamma(d+1)} = \frac{m}{d+m}\cdots\frac{1}{d+1}.$$

If $d > 0$ then $0 < B(-1, m-2) < 1$. Also,

$$\lim_{n\to\infty} B(-1, n-2) = 0.$$

We know according to (2.5)-(2.9)

$$\sum_{m=2}^{n-1} \frac{\Gamma(d+m)}{(m-1)!} = \frac{\Gamma(d+n)}{(d+1)(n-2)!} - \Gamma(d+1),$$

$$\sum_{m=1}^{n} \frac{\Gamma(d+m+1)}{m!} = \sum_{m=2}^{n+1} \frac{\Gamma(d+m)}{(m-1)!} = \frac{\Gamma(d+n+2)}{(d+1)\,n!} - \Gamma(d+1).$$

Hence, in case *a*= 1

$$\sum_{m=1}^{n} \frac{a^m}{B(-1, m-2)} = \sum_{m=1}^{n} \frac{1}{B(-1, m-2)} = \sum_{m=1}^{n} \frac{\Gamma(d+m+1)}{\Gamma(d+1)\,m!} =$$

$$\frac{1}{\Gamma(d+1)} \sum_{m=1}^{n} \frac{\Gamma(d+m+1)}{m!} = \frac{1}{\Gamma(d+1)} \left(\frac{\Gamma(d+n+2)}{(d+1)\,n!} - \Gamma(d+1) \right) = \frac{\Gamma(d+n+2)}{\Gamma(d+2)\,n!} - 1,$$

$$B(-1, n-2) \sum_{m=1}^{n} \frac{a^m}{B(-1, m-2)} = \frac{n!\Gamma(d+1)}{\Gamma(d+n+1)} \left(\frac{\Gamma(d+n+2)}{\Gamma(d+2)\,n!} - 1 \right) =$$

$$\frac{d+n+1}{d+1} - \frac{n!\Gamma(d+1)}{\Gamma(d+n+1)}.$$

We note that

$$\frac{n!}{\Gamma(d+n+1)} = \frac{n}{d+n} \frac{n-1}{d+n-1} \cdots \frac{1}{d+1} \frac{1}{\Gamma(d+1)},$$

$$\frac{n!\Gamma(d+1)}{\Gamma(d+n+1)} = \frac{n}{d+n} \frac{n-1}{d+n-1} \cdots \frac{1}{d+1}.$$

If *d* is a positive number, then

$$\lim_{n\to\infty} \frac{n!\Gamma(d+1)}{\Gamma(d+n+1)} = 0, \ \lim_{n\to\infty} \frac{d+n+1}{d+1} = \infty, \ \lim_{n\to\infty} B(-1, n-2) \sum_{m=1}^{n} \frac{a^m}{B(-1, m-2)} = \infty.$$

Hence, when $c \neq 0$

$$P(n) = B(-1, n-2) \left(c \sum_{m=1}^{n} \frac{a^m}{B(-1, m-2)} + 1 \right) =$$

$$cB(-1, n-2) \sum_{m=1}^{n} \frac{a^m}{B(-1, m-2)} + B(-1, n-2).$$

Hence in case *a* = 1

$$\lim_{n\to\infty} \left| B(-1, n-2) \left(c \sum_{m=1}^{n} \frac{a^m}{B(-1, m-2)} + 1 \right) \right| = \infty,$$

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} \frac{a^n c}{P(n)} = \lim_{n\to\infty} \frac{c}{P(n)} = 0.$$

Case 2: If we choose

$$G(-1) = ab^{-1}, G(l) = la + 1, \text{ for } n = 0, 1, 2, \ldots$$

then we obtain $G(l) \neq 0$ for all $l = -1, 0, 1, \ldots$. Also, for $m > 1$

$$B(-1, m-2) = ab^{-1} \prod_{l=0}^{m-2} (la+1) = ab^{-1} \frac{a^{m-1}\Gamma\left(m-1+\frac{1}{a}\right)}{a\Gamma\left(1+\frac{1}{a}\right)} =$$

$$\frac{a^{m-1}\Gamma\left(m-1+\frac{1}{a}\right)}{b\Gamma\left(1+\frac{1}{a}\right)},$$

$$\sum_{m=1}^{n} \frac{a^m}{B(-1, m-2)} = \sum_{m=1}^{n} \frac{ab\Gamma\left(1+\frac{1}{a}\right)}{\Gamma\left(m-1+\frac{1}{a}\right)},$$

$$B(-1, n-2) \sum_{m=1}^{n} \frac{a^m}{B(-1, m-2)} = \frac{a^{n-1}\Gamma\left(n-1+\frac{1}{a}\right)}{b\Gamma\left(1+\frac{1}{a}\right)} \sum_{m=1}^{n} \frac{ab\Gamma\left(1+\frac{1}{a}\right)}{\Gamma\left(m-1+\frac{1}{a}\right)} =$$

$$a^n \sum_{m=1}^{n} \frac{\Gamma\left(n-1+\frac{1}{a}\right)}{\Gamma\left(m-1+\frac{1}{a}\right)},$$

$$\frac{\Gamma\left(n-1+\frac{1}{a}\right)}{\Gamma\left(m-1+\frac{1}{a}\right)} = \frac{\left(n-2+\frac{1}{a}\right)\Gamma\left(n-2+\frac{1}{a}\right)}{\Gamma\left(m-1+\frac{1}{a}\right)} = \cdots = \frac{\left(n-2+\frac{1}{a}\right)\ldots\left(m-1+\frac{1}{a}\right)\Gamma\left(m-1+\frac{1}{a}\right)}{\Gamma\left(m-1+\frac{1}{a}\right)},$$

$$B(-1, n-2) \sum_{m=1}^{n} \frac{a^m}{B(-1, m-2)} = a^n \sum_{m=1}^{n} \prod_{l=2}^{n-m+1} \left(n-l+\frac{1}{a}\right) = \sum_{m=1}^{n} a^n \prod_{l=2}^{n-m+1} \left(n-l+\frac{1}{a}\right).$$

If $a \geq 1$ then $a^n \geq 1$ and

$$\lim_{n\to\infty} \frac{B(-1, n-2)}{a^n} \sum_{m=1}^{n} \frac{a^m}{B(-1, m-2)} = \infty,$$

$$\lim_{n\to\infty} \frac{B(-1, n-2)}{a^n} = \lim_{n\to\infty} \frac{a^{n-1}\Gamma\left(n-1+\frac{1}{a}\right)}{a^n b\Gamma\left(1+\frac{1}{a}\right)} = \lim_{n\to\infty} \frac{1}{ab}\left(n-2+\frac{1}{a}\right)\ldots(1+\frac{1}{a}) = \infty.$$

Hence, when $a \geq 1, c > 0$

$$P(n) = B(-1, n-2)\left(c\sum_{m=1}^{n} \frac{a^m}{B(-1, m-2)} + 1\right) =$$

$$cB(-1, n-2) \sum_{m=1}^{n} \frac{a^m}{B(-1, m-2)} + B(-1, n-2),$$

$$\lim_{n\to\infty} \frac{B(-1, n-2)}{a^n}\left(c\sum_{m=1}^{n} \frac{a^m}{B(-1, m-2)} + 1\right) = \infty,$$

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} \frac{c}{\frac{P(n)}{a^n}} = \lim_{n\to\infty} \frac{c}{\frac{B(-1,n-2)}{a^n}\left(c\sum_{m=1}^{n} \frac{a^m}{B(-1,m-2)} + 1\right)} = 0.$$

4 Two Systems With General Parameter r

We consider the following system just in case of positive initial values and positive value for r:

$$x_{n+1} = \frac{x_n}{x_{n-1}+r}, y_{n+1} = \frac{x_n y_n}{x_{n-1}y_{n-1}+r} \tag{4.1}$$

We will study the first equation at the begining since this equation is separated from the second one.

Lemma 4.1. *Suppose $x_{-1}, r > 0, x_0 = a > 0$. Then $x_n < ar^{-n}$ for $n = 1, 2, 3, \ldots$*

Proof. We start with $x_1 = \frac{x_0}{x_{-1}+r} = \frac{a}{x_{-1}+r} < \frac{a}{r} = ar^{-1}$ since $x_{-1}+r > r > 0$. We consider this relation as basis step. We continue by induction: Suppose that $x_k < ar^{-k}$ for some integer *k*. Then according to definition and that $x_{k-1} > 0$

$$x_{k+1} = \frac{x_k}{x_{k-1}+r} < \frac{x_k}{r} < \frac{ar^{-k}}{r} = \frac{a}{r^{k+1}}.\square$$

Lemma 4.2. *Assume $r, x_{-1}, y_{-1}, x_0, y_0 > 0$. Then the sequence y_n is positive.*

Proof. We start with

$$y_{0+1} = \frac{x_0 y_0}{x_{0-1}y_{0-1}+r} > 0, y_{1+1} = \frac{x_1 y_1}{x_0 y_0 + r} > 0.$$

We consider this relation as basis step. We continue by induction: Suppose that $y_k > 0$ for some integer *k*. Then, we obtain

$$y_{k+2} = \frac{x_{k+1}y_{k+1}}{x_k y_k + r} > 0$$

as the fraction of positive quantities. This is the induction step. $\square$

Lemma 4.3. *Assume that $r, x_{-1}, y_{-1} > 0$ and $x_0 = a > 0, y_0 = b$. Then*

$$y_n < a^n b r^{-0.5n(n+1)} \quad \text{for } n = 1, 2, 3, \dots$$

Proof. According to Lemma 4.2

$$y_1 = \frac{x_0 y_0}{x_{-1}y_{-1}+r} < \frac{ay_0}{r} = \frac{ab}{r}, y_2 = \frac{x_1 y_1}{x_0 y_0 + r} < \frac{x_1 y_1}{r} < \frac{1}{r}\frac{a}{r}\frac{ab}{r} = \frac{a^2 b}{r^3}.$$

We consider this relation as basis step. We denote by

$$f(n) = 1 + 2 + 3 + 4 + \cdots + n = \frac{n(n+1)}{2}.$$

Moreover,

$$y_3 < \frac{x_2 y_2}{r} < \frac{a}{r^3} y_2 < \frac{a}{r^3}\frac{a^2 b}{r^3} = \frac{a^3 b}{r^{f(3)}}.$$

We continue by induction: Suppose that

$$y_k < \frac{a^k b}{r^{f(k)}}$$

for some integer *k*. Then, we obtain

$$y_{k+1} = \frac{x_k y_k}{x_{k-1}y_{k-1}+r} < \frac{x_k y_k}{r} < \frac{a}{r^k}\frac{y_k}{r} = \frac{a}{r^{k+1}}\frac{a^k b}{r^{f(k)}} = \frac{a^{k+1}b}{r^{f(k)+k+1}} = \frac{a^{k+1}b}{r^{f(k+1)}}.\square$$

Proposition 4.4. *Assume that $r > 1$ and $x_{-1}, y_{-1}, x_0, y_0 > 0$. Then the sequence (x_n, y_n) converges to zero.*

Proof. According to Lemma 4.3

$$0 < x_n < \frac{x_0}{r^n} n = 1, 2, \dots$$

On the other hand

$$\frac{1}{r^n} \to 0 \quad \text{as} \quad n \to \infty.$$

We apply the Squeezing Theorem in order to conclude the result for x_n. Now, for y_n we know that according to Lemma 4.3

$$0 < y_n < \frac{{x_0}^n y_0}{r^{f(n)}}, n = 1, 2, \dots$$

Since $f(n) \to \infty$ as $n \to \infty$ we are done. □

We consider a special case, namely $r = 0$. In this case it is easy to compute the general solution. If we take the initial values

$$x_{-1} = a, x_0 = c, y_{-1} = b, y_0 = d$$

Then we obtain for $n = 1, 2, 3, \ldots$

$$x_{6n-2} = \frac{a}{c}, y_{6n-2} = \left(\frac{a}{c^2}\right)^{2n} \frac{cb}{d}, \quad x_{6n-1} = a, y_{6n-1} = \left(\frac{a^2}{c}\right)^{2n} d,$$

and for $n = 0, 1, 2, \ldots$

$$x_{6n} = c, y_{6n} = (ac)^{2n} d, \quad x_{6n+1} = \frac{c}{a}, y_{6n+1} = \left(\frac{c^2}{a}\right)^{2n} \frac{cd}{ab},$$

$$x_{6n+2} = \frac{1}{a}, y_{6n+2} = \frac{1}{b}\left(\frac{c}{a^2}\right)^{2n+1}, \quad x_{6n+3} = \frac{1}{c}, y_{6n+3} = \frac{1}{(ac)^{2n+1} d}.$$

We notice that we have a periodic solution, which consists of 6 elements. This is an essential change in the behavior of the sequence. It is an open problem, what will happen if *r* is negative.

We study now the system

$$x_{n+1} = \frac{x_{n-1}}{x_n + r}, \quad y_{n+1} = \frac{y_n}{x_{n-1} y_{n-1} + r} \tag{4.2}$$

We consider one vanishing initial value in order to simplify the calculations. For example, we set

$$x_{-1} = a, x_0 = 0, y_{-1} = b, y_0 = d \tag{4.3}$$

We obtain

$$x_1 = \frac{a}{r}, y_1 = \frac{d}{ab + r}, x_2 = \frac{0}{x_1 + 1} = 0, y_2 = \frac{\frac{d}{ab+r}}{r} = \frac{d}{abr + r^2},$$

$$x_3 = \frac{a}{r^2}, y_3 = \frac{\frac{d}{abr+r^2}}{\frac{a}{r}\frac{d}{(ab+r)} + r} = \frac{d}{ad + abr^2 + r^3},$$

$$x_4 = 0, y_4 = \frac{\frac{d}{ad+abr^2+r^3}}{r} = \frac{d}{adr + abr^3 + r^4},$$

$$x_5 = \frac{a}{r^3}, y_5 = \frac{\frac{d}{adr+abr^3+r^4}}{\frac{a}{r^2}\frac{d}{ad+abr^2+r^3} + r} = \frac{dr}{ad(r^0 + r^{0+3}) + abr^{2+3} + r^{3+3}},$$

$$y_6 = \frac{dr^1}{ad(r + r^4) + abr^6 + r^7}, y_7 = \frac{\frac{dr}{ad(r+r^4)+abr^6+r^7}}{\frac{a}{r^3}\frac{dr}{ad(1+r^3)+abr^5+r^6} + r} =$$

$$\frac{dr^{1+2}}{ad\left(r^1 + r^{1+3} + r^{1+6}\right) + abr^{5+4} + r^{6+4}}, y_8 = \frac{dr^{1+2}}{ad\left(r^2 + r^5 + r^8\right) + abr^{10} + r^{11}},$$

$$y_9 = \frac{dr^{1+2+3}}{ad\left(r^{1+2} + r^{1+2+3} + r^{1+2+6} + r^{1+2+9}\right) + abr^{9+5} + r^{10+5}} =$$

$$\frac{dr^{f(3)}}{ad\left(r^{f(2)} + r^{f(2)+3} + r^{f(2)+6} + r^{f(2)+9}\right) + abr^{f(5)-1} + r^{f(5)}}.$$

We use as previous the notation

$$f(n) = 1 + 2 + 3 + 4 + \cdots + n = \frac{n(n+1)}{2}.$$

Proposition 4.5. *Assume that $r \neq 1$. Then for $n > 2$ the solution of the system (4.2) and (4.3) is given by*

$$y_{2n+1} = \frac{dr^n(r^3-1)}{adr(r^{3n}-1) + abr^{3n}(r^3-1) + r^{3n+1}(r^3-1)}, y_{2n+2} = \frac{1}{r} y_{2n+1}.$$

Proof. According to previous calculations we see that for $n = 3, 4, 5, \ldots$

$$x_{2n} = 0, x_{2n+1} = \frac{a}{r^{n+1}},$$

$$y_{2n+1} = \frac{dr^{f(n-1)}}{adE(r,n) + abr^{f(n+1)-1} + r^{f(n+1)}},$$

$$y_{2n+2} = \frac{dr^{f(n-1)-1}}{adE(r,n) + abr^{f(n+1)} + r^{f(n+1)+1}},$$

where

$$E(r,n) = r^{f(n-2)} + r^{f(n-2)+3} + \cdots + r^{f(n-2)+3(n-1)}.$$

Hence

$$E(r,n) = r^{f(n-2)} \sum_{j=0}^{n-1} r^{3j} = r^{0.5(n-2)(n-1)} \frac{r^{3n}-1}{r^3-1},$$

$$y_{2n+1} = \frac{dr^{0.5n(n-1)}(r^3-1)}{adr^{0.5(n-2)(n-1)}(r^{3n}-1) + (abr^{0.5(n+1)(n+2)-1} + r^{0.5(n+1)(n+2)})(r^3-1)}.$$

After some simplifications we are done. □

Corollary 4.6. *If $ab + r, ad > 0$ and $|r| > 1$, then the solution of the system (4.2) and (4.3) converges to zero.*

Proof. We rewrite y_{2n+1} as follows

$$\frac{d(r^3-1)}{adr(r^{2n}-r^{-n}) + abr^{2n}(r^3-1) + r^{2n+1}(r^3-1)} = \frac{d(r^3-1)}{adr(r^{2n}-r^{-n}) + r^{2n}(r^3-1)(ab+r)}.$$

According to our assumptions r^{2n} tends to infinity, r^{-n} tends to zero. Further the denominator consists of two terms, which have the same sign. So its absolute value tends to infinity. Thus odd terms tend to zero, and the even terms do so, since they have less absolute value according to the formula. □

Finally, we turn our attention to system with general value for *r*. This system was studied first by Mashaqbeh (see [8]). We consider the system

$$x_{n+1} = \frac{x_n}{x_n + r} \quad , \quad y_{n+1} = \frac{x_{n-1} y_n}{x_{n-1} y_n + r} \tag{4.4}$$

with initial values

$$x_{-1} = a, x_0 = b, y_0 = c.$$

Proposition 4.7. *Let a, b, c and r be real numbers with $r \neq 1$. Then the solution of (4.4) is*

$$x_n = \frac{b(r-1)}{(b+r-1)r^n - b}, y_n = \frac{ac(1-r)^{n-1}}{(1+\frac{r-1}{b}; r)_{n-1} V(n)},$$

where $(a,q)_n$ gives the q-Pochhammer symbol and

$$V(n) = (-1)^{n-1} acU(n-3) + abcr^{n-2} + r^{n-1}(ac+r),$$

$$U(n) = \sum_{i=0}^{n} \frac{(r-1)^{n+2-i}(-r)^i}{(1+\frac{r-1}{b}; r)_{n+2-i}}.$$

Proof. We reach after tedious computations

$$x_n = \frac{b}{G(n)} = \frac{b}{b\frac{r^n-1}{r-1} + r^n} = -\frac{b(r-1)}{b - br^n - rr^n + r^n}.$$

Also,

$$H(n) = ac\sum_{i=0}^{n-3} b^{n-1-i}r^i\frac{B(n-2)}{B(n-2-i)} + abcr^{n-2}B(n-2) + r^{n-1}B(n-2)H(1) =$$

$$ab^{n-1}cB(n-2)\sum_{i=0}^{n-3}\frac{r^i}{b^iB(n-2-i)} + (abcr^{n-2} + r^{n-1}H(1))B(n-2) =$$

$$B(n-2)\left(ab^{n-1}c\sum_{i=0}^{n-3}\frac{r^i}{b^iB(n-2-i)} + abcr^{n-2} + r^{n-1}H(1)\right).$$

But

$$B(n) = \prod_{k=1}^{n} G(k) = \prod_{k=1}^{n}(b\frac{r^k-1}{r-1} + r^k) =$$

$$\prod_{k=1}^{n}\frac{(b+r-1)r^k - b}{(r-1)} = \frac{{}^{n}((b+r-1)r^k - b)}{(r-1)} = \frac{(-b)^{n+1}(1+\frac{r-1}{b};r)_{n+1}}{(r-1)^{n+1}},$$

So we reach

$$H(n) = \frac{(-b)^{n-1}(1+\frac{r-1}{b};r)_{n-1}}{(r-1)^{n-1}}(ab^{n-1}cT + abcr^{n-2} + r^{n-1}H(1)),$$

where

$$T = \sum_{i=0}^{n-3}\frac{r^i}{b^iB(n-2-i)} = \sum_{i=0}^{n-3}\frac{(r-1)^{n-i-1}r^i}{b^i(-b)^{n-1-i}(1+\frac{r-1}{b};r)_{n-1-i}} =$$

$$\sum_{i=0}^{n-3}\frac{(-1)^i(r-1)^{n-i-1}r^i}{(-b)^{n-1}(1+\frac{r-1}{b};r)_{n-1-i}} = (-b)^{1-n}W(n-3).$$

Thus,

$$H(n) = \frac{(-b)^{n-1}(1+\frac{r-1}{b};r)_{n-1}}{(r-1)^{n-1}}((-1)^{n-1}acW(n-3) + abcr^{n-2} + r^{n-1}H(1)),$$

$$y_n = \frac{ab^{n-1}c}{H(n)} = \frac{ab^{n-1}c(r-1)^{n-1}}{(-b)^{n-1}(1+\frac{r-1}{b};r)_{n-1}}\frac{1}{V(n)} = \frac{ac(1-r)^{n-1}}{(1+\frac{r-1}{b};r)_{n-1}V(n)}.\square$$

Remark 4.8. Let $a = b = c = 3$ and $r = 4$. The solution of (4.4) is according to the formula

$$\begin{pmatrix} x_3 \\ y_3 \end{pmatrix} = \begin{pmatrix} \frac{3}{127} \\ \frac{81}{2293} \end{pmatrix}, \begin{pmatrix} x_4 \\ y_4 \end{pmatrix} = \begin{pmatrix} \frac{3}{511} \\ \frac{243}{284575} \end{pmatrix}, \begin{pmatrix} x_5 \\ y_5 \end{pmatrix} = \begin{pmatrix} \frac{3}{2047} \\ \frac{729}{144564829} \end{pmatrix},$$

$$V(3) = \frac{81}{7} + 108 + 208 = \frac{2293}{7}, W(0) = \sum_{i=0}^{0}\frac{(4-1)^{2-i}(-4)^i}{(1+\frac{4-1}{3};4)_{2-i}} = \frac{3^2(-4)^0}{(2;4)_2} = \frac{9}{7},$$

$$H(3) = ab^2c + rabcG(1) + r^2G(1)H(1) = 81 + 4 * 27 * 7 + 16 * 7 * 13 = 2293,$$

$$H(3) = \frac{(-b)^2(1+\frac{r-1}{b};r)_2}{(r-1)^2}(acW(0) + abcr + r^2H(1)) = \frac{(-3)^2(2;4)_2}{3^2}$$

$$\left(9 * \frac{9}{7} + 27 * 4 + 16 * 13\right) = 7\left(\frac{81}{7} + 27 * 4 + 16 * 13\right) = 81 + 756 + 1456 = 2293,$$

$$y_4=\frac{9(-3)^3}{(2;4)_3V(4)}=\frac{-9*27}{-217*\frac{284575}{217}}=\frac{243}{284575},$$

$$V(4)=(-1)^3 9W(1)+27*16+64*13=-9*(-\frac{1143}{217})+1264=\frac{284\,575}{217},$$

$$W(1)=\sum_{i=0}^{1}\frac{3^{3-i}(-4)^i}{(2;4)_{3-i}}=\frac{3^3}{(2;4)_3}+\frac{3^2(-4)}{(2;4)_2}=\frac{27}{-217}+\frac{-36}{7}=-\frac{1143}{217},$$

$$H(4)=\frac{(-3)^3(2;4)_3}{(3)^3}((-1)^3 9W(1)+27*16+64*13)=-(-217)(-9W(1)+27*16+64*13)=$$

$$217(-9*(-\frac{1143}{217})+27*16+64*13)=284575,$$

$$H(4)=243+4*G(2)*H(3)=243+4*G(2)*2293=243+4*(3+4*7)*2293=284575,$$

$$W(2)=\sum_{i=0}^{2}\frac{(3)^{4-i}(-4)^i}{(2;4)_{4-i}}=\frac{3^4}{(2;4)_4}+\frac{3^3*(-4)}{(2;4)_3}+\frac{3^2(16)}{(2;4)_2}=\frac{81}{27559}+\frac{-108}{-217}+\frac{144}{7}=\frac{580725}{27559},$$

$$V(5)=(-1)^{5-1}9W(5-3)+27*4^{5-2}+4^{5-1}(13)=\frac{5226525}{27559}+1728+3328=\frac{144564829}{27559},$$

$$y_5=\frac{3*3^{5-1}*3}{H(5)}=\frac{3*3^{5-1}*3(4-1)^{5-1}}{(-3)^{5-1}(1+\frac{4-1}{3};4)_{5-1}}\frac{1}{V(5)}=\frac{59049}{81*27559*\frac{144564829}{27559}}=\frac{729}{144564829}$$

or

$$H(5)=3*3^4*3+4G(3)H(4)=729+4*127*284575=144564829,$$

$$y_5=\frac{3*3^{5-1}*3}{H(5)}=\frac{729}{144564829}$$

On the other hand

$$\begin{pmatrix}x_1\\ y_1\end{pmatrix}=\begin{pmatrix}\frac{x_0}{x_0+4}\\ \frac{x_{-1}y_0}{x_{-1}y_0+4}\end{pmatrix}=\begin{pmatrix}\frac{3}{3+4}\\ \frac{3*3}{3*3+4}\end{pmatrix}\begin{pmatrix}\frac{3}{7}\\ \frac{9}{13}\end{pmatrix}$$

$$\begin{pmatrix}x_2\\ y_2\end{pmatrix}=\begin{pmatrix}\frac{x_1}{x_1+4}\\ \frac{x_0y_1}{x_0y_1+4}\end{pmatrix}=\begin{pmatrix}\frac{\frac{3}{7}}{\frac{3}{7}+4}\\ \frac{3*\frac{9}{13}}{3*\frac{9}{13}+4}\end{pmatrix}=\begin{pmatrix}\frac{3}{31}\\ \frac{27}{79}\end{pmatrix}$$

$$\begin{pmatrix}\boldsymbol{x}_3\\ \boldsymbol{y}_3\end{pmatrix}=\begin{pmatrix}\frac{\boldsymbol{x}_2}{\boldsymbol{x}2+4}\\ \frac{\boldsymbol{x}_1\boldsymbol{y}_2}{\boldsymbol{x}_1\boldsymbol{y}_2+4}\end{pmatrix}=\begin{pmatrix}\frac{\frac{3}{31}}{\frac{3}{31}+4}\\ \frac{\frac{3}{7}*\frac{27}{79}}{\frac{3}{7}*\frac{27}{79}+4}\end{pmatrix}=\begin{pmatrix}\frac{3}{127}\\ \frac{81}{2293}\end{pmatrix}$$

$$\begin{pmatrix}x_4\\ y_4\end{pmatrix}=\begin{pmatrix}\frac{x_3}{x_3+4}\\ \frac{x_2y_3}{x_2y_3+4}\end{pmatrix}=\begin{pmatrix}\frac{\frac{3}{127}}{\frac{3}{127}+4}\\ \frac{\frac{3}{31}*\frac{81}{2293}}{\frac{3}{31}*\frac{81}{2293}+4}\end{pmatrix}=\begin{pmatrix}\frac{3}{511}\\ \frac{243}{284575}\end{pmatrix}$$

$$\begin{pmatrix}x_5\\ y_5\end{pmatrix}=\begin{pmatrix}\frac{x_4}{x_4+4}\\ \frac{x_3y_4}{x_3y_4+4}\end{pmatrix}=\begin{pmatrix}\frac{\frac{3}{511}}{\frac{3}{511}+4}\\ \frac{\frac{3}{127}*\frac{243}{284575}}{\frac{3}{127}*\frac{243}{284575}+4}\end{pmatrix}=\begin{pmatrix}\frac{3}{2047}\\ \frac{729}{144564829}\end{pmatrix}$$

Conclusions

We determined the limit of some sequences without determined the explicit formula of the solution, which might be not easily expressible in closed form. We encountered many formulas, which were given by the software Mathematica. This was in the case $r=1$. There are still cases to be studied in the future since we studied special cases due to the lack of formulas of summations in some cases. In one of the systems we set one initial value zero in order to simplify calculations. We can analogously set other initial values to zero. But, the general case of arbitrary initial values is more likely to be complicated. This case can be a subject for future studies. Also the explicit formula of the solution was calculated in case $r\neq 1$. But, the limit was determined based upon this knowledge for $|r|>1$ for some equations, while for other equations this was not possible since we do not have enough information about the Pochammer symbol.

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