Palestine Journal of Mathematics

Vol. 11(Special Issue II)(2022) , 158–164

ISO-C-RETRACTABLE MODULES AND RINGS

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Communicated by Nouressaat Touafek

MSC 2010 Classifications: Primary 16D10, 16D50; Secondary 16D80.

Keywords and phrases: Iso-c-retractable modules, Iso-retractable modules, C-retractable modules, Extending modules.

The authors are thankful to the referee for his/her useful comments to improve this paper. The research of the second author was supported by a grant from University Grants Commission, India (UGC Ref. No.: 1211(CSIR-UGC NET DEC. 2018)).

Abstract The classes of retractable and extending modules lie strictly between the classes of iso-retractable modules and c-retractable modules. We introduce a notion of iso-c-retractable modules which also lies strictly between these classes. In support, we give some examples. We give characterizations of iso-c-retractable modules, iso-retractable modules and simple modules. Further, we prove that a module is iso-retractable if and only if it is essentially iso-retractable and iso-c-retractable. We prove that an iso-c-retractable module is continuous if and only if GQ-injective if and only if it satisfies C_2-condition; and self-c-injective if and only if it satisfies C_1-condition. Also, an iso-c-retractable module satisfies C_1-condition if it is d-Rickart or quasi principally injective or hereditary. As a consequence, we find that a d-Rickart module, hereditary module, quasi-injective module and self-c-injective module are uniform if and only if it is iso-c-retractable and indecomposable.

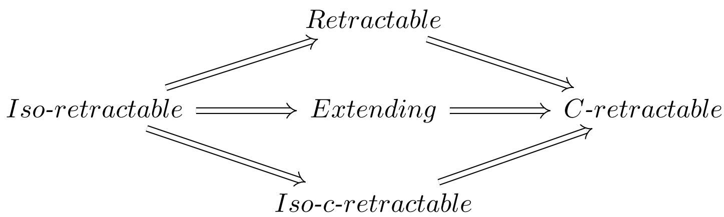
1 Introduction

Throughout this paper, all rings are associative with identity 1 and all modules are right unital modules unless otherwise stated. The readers are referred to [10] for all undefined terminologies and notions.

Recall [8], let K and N be two submodules of a module M. Then, submodule K is called a *complement* of N (in M) if, K is maximal in the collection of submodules Q of M such that $Q \cap N = 0$. A submodule C of a module M is called a *complement submodule* if, it is a complement of some submodule of M. A module M is called *extending* if every complement submodule is a direct summand.

Following [1], a module M is called *compressible* if, for each nonzero submodule N of M, there exists a monomorphism M to N. In 1979, Khuri [9] defined the concept of retractable modules as a generalization of compressible modules. He called a module M *retractable* if, for each nonzero submodule N of M, there exists a nonzero homomorphism M to N. In 1980, Chatters and Khuri defined the concept of c-retractable modules as a generalization of retractable modules. They call a module M *c-retractable* if, for any nonzero complement submodule C of M, there exists a nonzero homomorphism M to C. Recently, first author defined the notion of iso-retractable modules in [3, 4] which is properly contained in the classes of retractable and compressible modules. He calls a module M *iso-retractable* if for each nonzero submodule N of M, there exists an isomorphism M to N. There are some other generalizations of above concepts.

By the motivation of above concepts, we introduce a notion of iso-c-retractable modules which lies strictly between the classes of c-retractable modules and iso-retractable modules. Since, every iso-retractable module is extending (see [7, Theorem 1.12]) and every extending module is c-retractable, we have the following diagram:



In Section 2, we give definition and examples of iso-c-retractable modules. We give a characterization of iso-c-retractable rings and iso-c-retractable modules (see Proposition 2.3 and 2.4). Also, we discuss some basic properties of it(see Corollary 2.5, Remark 2.6, Proposition 2.7 and 2.8). Further, we give a characterization of simple modules and iso-retractable modules (see Proposition 2.9 and Theorem 2.11).

In Section 3, we discuss the relation of iso-c-retractable modules with injective and projective modules. We find some equivalent classes of modules over the class of iso-c-retractable modules (see Proposition 3.2 and Theorem 3.4). Also, we give some sufficient conditions over which iso-c-retractable modules are extending (see Theorem 3.5).

2 Iso-c-retractable modules and rings

Definition 2.1. We call a module M iso-c-retractable if for every nonzero complement submodule N of M, there exists an isomorphism $f : M \to N$.

We call a ring R right (respectively, left) iso-c-retractable if R_R (respectively, ${}_RR$) is iso-c-retractable. Naturally, a ring R is called iso-c-retractable if it is both left and right iso-c-retractable.

Example 2.2. (i) Every iso-retractable module is iso-c-retractable. However, its converse need not be true in general. For example, if p is a prime number, then $\mathbb{Z}_{p^2}$ as a $\mathbb{Z}$-module is iso-c-retractable while it is not iso-retractable.

(ii) Every iso-c-retractable module is c-retractable. However, its converse need not be true in general. For example, if p and q are distinct prime numbers, then $\mathbb{Z}_{pq}$ as a $\mathbb{Z}$-module is c-retractable while it is not iso-c-retractable.

Proposition 2.3. *A ring R is right iso-c-retractable if and only if for every complement right ideal I of R, $\exists a \in Reg(R)$ such that $I = aR$.*

Proof. It follows from the fact that a right ideal I is isomorphic to R if and only if $I = aR$ for some $a \in Reg(R)$. □

Proposition 2.4. *The following are equivalent for a module M:*

(i) M is iso-c-retractable;

(ii) There is an iso-c-retractable module M' and a monomorphism $f : M \to M'$ such that $f(M) \leq_c M'$.

(iii) M is isomorphic to an iso-c-retractable module.

Proof. (1) $\Longrightarrow$ (2). Take $M' = M$ and $f = I_M$, then it is clear.

(2) $\Longrightarrow$ (3). Suppose that there is an iso-c-retractable module M' and a monomorphism $f : M \to M'$ such that $f(M) \leq_c M'$. If $f(M) = 0$, then $M = 0$ and we are done. Suppose $f(M) \neq 0$. Then, $f(M) \cong M'$ as M' is iso-c-retractable. Therefore $M \cong f(M) \cong M'$.

(3) $\Longrightarrow$ (1). Clear. □

Corollary 2.5. *Complement submodules of every iso-c-retractable module are iso-c-retractable. In particular, direct summands of every iso-c-retractable module are iso-c-retractable.*

Remark 2.6. Direct sum of two iso-c-retractable module need not be iso-c-retractable. For example, $\{\bar{0}, \bar{3}\}$ and $\{\bar{0}, \bar{2}, \bar{4}\}$ are iso-c-retractable submodules of $\mathbb{Z}_6$ as $\mathbb{Z}$-module such that $\mathbb{Z}_6 = \{\bar{0}, \bar{3}\} \oplus \{\bar{0}, \bar{2}, \bar{4}\}$ while $\mathbb{Z}_6$ as $\mathbb{Z}$-module is not iso-c-retractable.

In general, quotient (homomorphic image) of an iso-c-retractable module need not be iso-c-retractable. For example, $\mathbb{Z}$ as $\mathbb{Z}$ module is iso-c-retractable but $\mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}_6$ as $\mathbb{Z}$ module is not iso-c-retractable. However, we observe the following:

Proposition 2.7. *Let M be an iso-c-retractable module. If C is a complement submodule of M such that $f(C) + f^{-1}(C) \subseteq C$ or $f(C) = C$, for every injective endomorphism f of M, then M/C is iso-c-retractable.*

Proof. Let $\bar{0} \neq K/C \leq_c M/C$. Then, $0 \neq K \leq_c M$ and so there exists an isomorphism $f : M \to K$, as M is iso-c-retractable. Hence, $f : M \to M$ is a monomorphism such that $Im(f) = K$. Therefore, by the hypothesis, $f(C) + f^{-1}(C) \subseteq C$ or $f(C) = C$. Define a map $\overline{f} : M/C \to K/C$ by $\overline{f}(x + C) = f(x) + C; \forall x + C \in M/C$. Then, $\overline{f}$ is a well defined homomorphism as $f(C) \subseteq C$ in both cases. Let $y + C \in K/C$. Then $y \in K$ and so $\exists m \in M$ such that $f(m) = y$, as $f : M \to K$ is surjective. Hence, $y + C = f(m) + C = \overline{f}(m + C)$. Thus $\overline{f}$ is surjective.

Case-I: Suppose that $f(C) + f^{-1}(C) \subseteq C$. Then, $\overline{f}(x + C) = C \implies f(x) + C = C \implies f(x) \in C \implies x \in f^{-1}(C) \subseteq f(C) + f^{-1}(C) \subseteq C$. It follows that $\overline{f}$ is a monomorphism.

Case-II: Suppose that $f(C) = C$. Then, $\overline{f}(x+C) = C \implies f(x)+C = C \implies f(x) \in C$. Now, since $f(x) \in C$ and $C = f(C)$, $\exists c \in C$ such that $f(x) = f(c)$ which implies that $x - c \in ker(f) = 0$ and so $x = c \in C$. Hence, $\overline{f}$ is a monomorphism.

Thus, $\overline{f}$ is an isomorphism. □

Proposition 2.8. *If M is a uniform module, then M is iso-c-retractable. The converse holds if M has finite uniform dimension.*

Proof. Since uniform modules have no nonzero proper complement submodule, it is clear that every uniform module is iso-c-retractable. Conversely, suppose that M is iso-c-retractable and has finite uniform dimension. Let K be a nonzero complement submodule of M. Then there exists an isomorphism $f : M \to K$. Hence, we have a monomorphism $f_1 = iof : M \to M$ where $i : K \to M$ is the inclusion map. Since M has finite uniform dimension, by [8, 5.8(4)], $Im(f_1) = K$ is an essential submodule of M which implies that $K = M$. Thus, M has no nonzero proper complement submodule and so M is uniform. □

Proposition 2.9. *The following are equivalent for a module M:*

(i) M is simple;

(ii) M is nonzero semisimple and iso-retractable;

(iii) M is nonzero semisimple and iso-c-retractable.

Proof. (1) $\implies$ (2) $\implies$ (3). Clear

(3) $\implies$ (1). Since M is nonzero semisimple, M has a simple submodule, say, S which will be a complement submodule of M. Since M is iso-c-retractable, $S \cong M$. Thus, M is simple. □

Recall [5], a module M is *essentially iso-retractable* if every essential submodule is isomorphic to M.

Lemma 2.10. *[14, Proposition 2.5] For any submodule N of a module M, there exists a submodule K such that $N \leq_e K \leq_c M$.*

Theorem 2.11. *The following are equivalent for a module M:*

(i) M is iso-retractable;

(ii) M is essentially iso-retractable and uniform;

(iii) M is essentially iso-retractable and iso-c-retractable.

Proof. (1) $\implies$ (2). It follows from [4, Theorem 1.12].

(2) $\implies$ (3). Clear.

(3) $\implies$ (1). Let N be a nonzero submodule of M. Then, by Lemma 2.10, there exists a submodule K such that $N \leq_e K \leq_c M$. Since M is iso-c-retractable and K is a nonzero complement submodule of M, $K \cong M$. Now, since M is essentially iso-retractable and $K \cong M$, K is is essentially iso-retractable by [5, Proposition 2.6]. It follows that $N \cong K \cong M$. □

3 Some variants of injectivity and Iso-c-retractable modules

Recall [12], a module M is called *directly finite* if M is not isomorphic to any proper summand of itself. A module P is called *purely infinite* if $P \cong P \oplus P$. Two modules are called *orthogonal* if they have no nonzero isomorphic submodule. Note that any decomposable iso-c-retractable module cannot be directly finite.

Proposition 3.1. *Every injective iso-c-retractable module is either directly finite or purely infinite.*

Proof. Let M be an injective iso-c-retractable module. Since M is injective, by [12, Theorem 1.35], $M = D \oplus P$ where D is directly finite and P is purely infinite module such that D and P are orthogonal. If possible, suppose that D and P both are nonzero. Then, $P \cong M \cong D$ because M is iso-c-retractable. Which gives a contradiction to the fact that D and P are orthogonal. Hence, only one of D and P is nonzero. It follows that either $M = D$ or $M = P$. This completes the proof. □

Recall [12], consider the following statements for a module M:

(C_1) : every complement submodule M is a direct summand of M.

(C_2) : every submodule of M which is isomorphic to a direct summand of M is itself a direct summand of M.

A module satisfying C_1-condition is called *extending* or *CS*; and a module satisfying C_1 and C_2-condition is called *continuous*. Recall [2], a module M is called (*generalized Quasi-injective*) *GQ-injective* if for any submodule N isomorphic to a complement submodule of M, any homomorphism N to M can be extended to M.

Proposition 3.2. *The following are equivalent for an iso-c-retractable module M:*

(i) M is continuous;

(ii) M is GQ-injective;

(iii) M satisfies C_2-condition.

Proof. (1) $\Longrightarrow$ (2). Follows from [2, Corollary 1].

(2) $\Longrightarrow$ (3). Follows from [2, Lemma 1].

(3) $\Longrightarrow$ (1). Suppose M satisfies C_2-condition. Let C be a nonzero complement submodule of M. Then, $C \cong M$ because M is iso-c-retractable. Since M is a direct summand of M and M satisfies C_2-condition, C is a direct summand of M. □

Recall [13], let M_1 and M_2 be modules. The module M_2 is M_1-*c-injective* if, every homomorphism $\alpha : K \to M_2$, where K is a complement submodule of M_1, can be extended to a homomorphism $\beta : M_1 \to M_2$. A module M is *self-c-injective* if it is M-c-injective.

Lemma 3.3. *[13, Lemma 2.1] K be a complement submodule of a module M. If K is M-c-injective then K is a direct summand.*

Theorem 3.4. *The following are equivalent for an iso-c-retractable module M:*

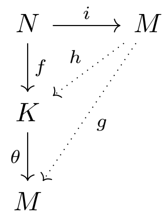
(i) M is self-c-injective;

(ii) every complement submodule is M-c-injective;

(iii) M is extending;

(iv) every module is M-c-injective.

Proof. (1) $\Longrightarrow$ (2). Let K be a complement submodule of M. If K is zero, we are done. Suppose that K is nonzero. Then there exists an isomorphism $\theta : K \to M$. If N is a complement submodule of M and $f : N \to K$ is a nonzero homomorphism then we have the following diagram:



Since M is self-c-injective, there exists $g : M \to M$ such that $goi = \theta of$. If we take $h := \theta^{-1} og : M \to K$ then $hoi = f$. Hence K is M-c-injective.

(2) $\Longrightarrow$ (3). Let K be a complement submodule of M. Then, by hypothesis, K is M-c-injective and so K is a direct summand of M by Lemma 3.3. Thus M is extending.

(3) $\Longrightarrow$ (4) $\Longrightarrow$ (1). Clear. □

Recall [11], a module M is called *d-Rickart* (or *dual Rickart*) if, for every $f \in End_R(M)$, $Im(f)$ is a direct summand of M.

Theorem 3.5. *Let M be an iso-c-retractable module. Then, M is extending if M satisfies any one of the following three conditions:*

(i) M is hereditary.

(ii) M is d-Rickart.

(iii) M is quasi-principally injective.

Proof. (1). Suppose that M is hereditary and let K be a nonzero complement submodule of M. Then, there exists an isomorphism $\theta : M \to K$. Since M is hereditary, K is projective. Hence, there exists a homomorphism $h : K \to M$ such that $\theta oh = I_K$ which implies that K is a direct summand of M. Thus, M is extending.

(2). Suppose that M is d-Rickart and let C be a nonzero complement submodule of M. Then, there exists an isomorphism $f : M \to C$. This implies that $f : M \to M$ is a monomorphism such that $Im(f) = C$. Since M is d-Rickart, $Im(f) = C$ is a direct summand of M. Thus, M is extending.

(3). Suppose that M is quasi-principally injective and let C be a nonzero complement submodule of M. Then, there exists an isomorphism $f : M \to C$. Hence, we have a monomorphism $f : M \to M$ such that $f(M) = C$. Since M is quasi-principally injective, $f(M) = C$ is a direct summand of M by [6, Lemma 4.6]. Thus, M is extending. □

Proposition 3.6. *Let M be a module satisfying any one of the following conditions:*

(i) M is hereditary.

(ii) M is d-Rickart.

(iii) M is quasi-principally injective.

(iv) M is self-c-injective.

Then, M is uniform if and only if it is iso-c-retractable and indecomposable.

Proof. Suppose that M is iso-c-retractable and indecomposable. If M satisfies any one condition of the Proposition, then, by Theorem 3.4 and 3.5, M is an indecomposable extending module and hence M is uniform. The converse is clear. □

Recall [8], the *second singular submodule* of a module M is denoted by $Z_2(M)$ and defined as $Z(M/Z(M)) = Z_2(M)/Z(M)$, where $Z(K)$ denotes the singular submodule of the module K. A module M is called *Z_2-torsion* if $Z_2(M) = M$.

Theorem 3.7. *Let M be an iso-c-retractable module for which $Z_2(M)$ is nonzero proper. Then, the following are equivalent:*

(i) M is quasi-injective;

(ii) M is quasi-principally injective;

(iii) M is self-c-injective;

(iv) M is extending.

Proof. (1) $\Longrightarrow$ (2) and (1) $\Longrightarrow$ (3) are clear.

(2) $\Longrightarrow$ (4). follows from Theorem 3.5.

(3) $\Longrightarrow$ (4). follows from Theorem 3.4.

(4) $\Longrightarrow$ (1). Suppose that M is extending. Then, by [8, 7.11], $M = Z_2(M) \oplus M'$ such that $Z_2(M)$ and M' both are extending and $Z_2(M)$ is M'-injective. Since $Z_2(M)$ is nonzero proper, it follows that $Z_2(M)$ and M' are nonzero complement submodules of M; and so $M \cong M'$ and $M \cong Z_2(M)$ as M is iso-c-retractable. Thus, M is M-injective, i.e., M is quasi-injective. □

Recall [10], a ring R is semisimple if and only if every simple R-module is projective; and a ring R is a V-ring if and only if every simple R-module is injective.

Proposition 3.8. *Let R be a ring.*

(i) If every iso-c-retractable R-module is projective, then R is a semi-simple ring.

(ii) If every iso-c-retractable R-module is injective, then R is a V-ring.

Proof. (1). Let M be a simple R-module. Then M is iso-c-retractable and so projective by the assumption. Thus, every simple R-module is projective and so R is semi-simple.

(2). Let M be a simple R-module. Then M is iso-c-retractable and so M is injective by the assumption. Thus, every simple R-module is injective and so R is a V-ring. □

Corollary 3.9. *[3, Proposition 2.4, Corollary 2.9] Let R be a ring.*

(i) If every iso-retractable R-module is projective, then R is a semi-simple ring.

(ii) If every iso-retractable R-module is injective, then R is a V-ring.

Remark 3.10. (i) Since torsion submodule of any module over a commutative domain is a complement submodule [10, Example 6.34], every iso-c-retractable module over a commutative domain is either torsion or torsion-free.

(ii) Since second singular submodule is always a complement submodule [14, Exercise 2.10(iii)], every iso-c-retractable module is either Z_2-torsion or nonsingular.

(iii) Since every injective submodule is a direct summand, a nonzero iso-c-retractable module is injective if and only if it has a nonzero injective submodule.

(iv) Since isomorphisms are preserved by Morita equivalent rings, iso-retractability is a Morita equivalent property.

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Received: June 18, 2021.

Accepted: December 22, 2021.