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Continuity of Index Kontorovich-Lebedev transform on certain function space and associated convolution operator

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Abstract In this paper, a brief study about index Kontorovich-Lebedev transform (index KL-transform) has been carried out. Translation and convolution operator associated to it have been defined. Estimates for translation and convolution operators in Lebesgue space are obtained. Further, continuity of index KL-transform on certain function space is discussed.

1 Introduction

In this article, study related to an index Kontorovich-Lebedev transform (index KL-transform) has been carried out. Kontorovich-Lebedev transform first studied by Kontorovich and Lebedev [1], which contains modified Bessel function or MacDonald function $K_{i\tau}(x)$, with purely imaginary index in its kernel as

$$(KLf)(\tau) = \int_0^\infty K_{i\tau}(x)f(x)dx,$$

and then various researcher have carried the research in this area with different approach. As Hankel transform with various modification in the kernel function, that is in Bessel function of first kind, has been defined in several ways [2, 3]. Similarly, with some modifications in the kernel, KL-transform has been defined in several ways and various studies related to it have been done [4, 5, 7, 8, 6]. Yakubovich is one of the pioneers' in this field and has done several work related to KL-transform and its modified forms.
Yakubovich [6, p. 691], considered the definition of index KL-transform as

$$(\mathsf{K}\varphi)(x) = 2\int_0^\infty \tau K_{i\tau}(2\sqrt{x})\varphi(\tau)d\tau, \tag{1.1}$$

where $K_{i\tau}(2\sqrt{x})$ represents modified Bessel function with purely imaginary index.
From [4, p. 14], its integral representation is given as

$$K_{i\tau}(2\sqrt{x}) = \int_0^\infty e^{-2\sqrt{x}\cosh(t)}\cos(\tau t)dt. \tag{1.2}$$

Inversion of (1.1), is given as

$$\varphi(\tau) = \frac{1}{2\pi^2}\sinh(\pi\tau)\int_0^\infty K_{i\tau}(2\sqrt{x})(\mathsf{K}\varphi)(x)\frac{dx}{x}. \tag{1.3}$$

The modified Bessel function $K_{i\tau}(2\sqrt{x})$ is an eigenfunction of differential operator

$$\mathcal{A}_x = x^2D_x^2 + xD_x - x, \tag{1.4}$$

where $D_x = \frac{d}{dx}$. Moreover, we have

$$\mathcal{A}_x K_{i\tau}(2\sqrt{x}) = -\frac{\tau^2}{4} K_{i\tau}(2\sqrt{x}). \tag{1.5}$$

This can be easily verified using recurrence relation for modified Bessel function. The series representation of $\mathcal{A}_x^n$ is given as

$$\mathcal{A}_x^n = \sum_{j=0}^{2n} x^j P_j^n(x) D_x^j, \ \forall n \in \mathbb{N}_0, \tag{1.6}$$

where the P_j^n are polynomials of degree $n - \frac{j}{2}$ for even j and $n - \frac{(j+1)}{2}$ for odd j respectively. Further, the adjoint of index KL-transform (1.1), can be written as

$$(\mathcal{K}\varphi)(\tau) = 2\tau \int_0^\infty K_{i\tau}(2\sqrt{x})\varphi(x)dx, \tag{1.7}$$

and the corresponding inversion of (1.7), is

$$\varphi(x) = \frac{1}{2\pi^2 x} \int_0^\infty K_{i\tau}(2\sqrt{x}) \sinh(\pi\tau)(\mathcal{K}\varphi)(\tau)d\tau. \tag{1.8}$$

Using the relation 1.103 from [4, p. 15], we can write

$$\begin{aligned} K_{i\tau}(2\sqrt{x})K_{i\tau}(2\sqrt{y}) &= \frac{1}{4}\int_0^\infty \exp\Big(-\Big(\frac{xy+yz+zx}{\sqrt{xyz}}\Big)\Big)K_{i\tau}(2\sqrt{z})\frac{dz}{z} \\ &= \int_0^\infty K(x,y,z)K_{i\tau}(2\sqrt{z})dz, \end{aligned} \tag{1.9}$$

where

$$K(x,y,z) = \frac{1}{4z}\exp\Big(-\Big(\frac{\mathrm{xy}+\mathrm{yz}+\mathrm{zx}}{\sqrt{\mathrm{xyz}}}\Big)\Big). \tag{1.10}$$

From (1.10), we can see that $K(x,y,z)$ is symmetric in $x,\ y$ but not in z.
From [6, p. 692], Parseval's relation for index KL-transform (1.1), is given as

$$\int_0^\infty |(\mathsf{K}\varphi)(x)|^2 \frac{dx}{x} = 4\pi^2 \int_0^\infty \frac{\tau}{\sinh(\pi\tau)}|\varphi(\tau)|^2 d\tau.$$

In the similar manner, Parseval's relation for index KL-transform (1.7), can be written as

$$\int_0^\infty |(\mathcal{K}\varphi)(\tau)|^2 \frac{\sinh(\pi\tau)}{\tau} d\tau = 4\pi^2 \int_0^\infty |\varphi(x)|^2 x\, dx.$$

2 Preliminary results and convolution operator

In this section, we obtain some useful results that will play an important role in investigation of estimates of convolution operator and continuity of the index KL-transform.
From [9, p. 344], we have

$$\int_0^\infty w^{\alpha-1}e^{-uw-\frac{v}{w}}dw = 2\Big(\frac{v}{u}\Big)^{\frac{\alpha}{2}} K_\alpha(2\sqrt{uv}), \qquad Re(u),\ Re(v) > 0,\ \alpha \in \mathbb{R}.$$

Putting $u = \frac{x+y}{\sqrt{xy}}$, $v = \sqrt{xy}$ and $w = \sqrt{z}$, we get

$$\int_0^\infty z^{\frac{\alpha}{2}-1} e^{-\frac{xy+yz+zx}{\sqrt{xyz}}} dz = 4\Big(\frac{xy}{x+y}\Big)^{\frac{\alpha}{2}} K_\alpha(2\sqrt{x+y})$$

In view of (1.10) and for $\alpha = 0$, above equation reduces to

$$\int_0^\infty K(x,y,z)dz = K_0(2\sqrt{x+y}). \tag{2.1}$$

From [10, p. 97(69)], an integral representation of modified Bessel function is

$$[K_{i\tau}(2\sqrt{x})]^2 \sinh(\pi\tau) \;=\; \pi \int_0^\infty J_0(4\sqrt{x}\sinh(t))\sin(2\tau t)dt.$$

From [8, p. 32], we have

$$|J_0(x)| \;\leq\; C(b)x^{-b},$$

where $0 < b < 1/2$. Thus

$$|[K_{i\tau}(2\sqrt{x})]^2 \sinh(\pi\tau)| \;\leq\; C(b)\pi(4\sqrt{x})^{-b} \int_0^\infty [\sinh(t)]^{-b}dt$$

involved integral is convergent for $0 < b < 1/2$. Hence

$$|\; K_{i\tau}(2\sqrt{x}) \;| \;\leq\; C'(b)\, \frac{x^{\frac{-b}{4}}}{\sqrt{\sinh(\pi\tau)}}. \tag{2.2}$$

For $n \in \mathbb{N}$, using (1.2), we have

$$D_x^n K_{i\tau}(2\sqrt{x}) \;=\; \sum_{r=1}^n C(-1)^{n-2j} x^{\frac{r-2n}{2}} \int_0^\infty (\cosh(t))^r e^{-2\sqrt{x}\cosh(t)} \cos(\tau t)dt,$$

where C is positive constant. It can be estimated as

$$\begin{aligned} |D_x^n K_{i\tau}(2\sqrt{x})| \;&\leq\; \sum_{r=1}^n C\, x^{\frac{r-2n}{2}} \int_0^\infty e^{tr}\, e^{-\sqrt{x}e^t} dt = \sum_{r=1}^n C\, x^{\frac{r-2n}{2}} \int_0^\infty u^{r-1}\, e^{-\sqrt{x}u} du \\ &=\; \sum_{r=1}^n C\, x^{\frac{r-2n}{2}}\, \frac{(r-1)!}{x^{\frac{r}{2}}} \leq C'\, x^{-n}, \end{aligned} \tag{2.3}$$

where C' is positive constant. Again from (1.2), we can write

$$D_\tau^n K_{i\tau}(2\sqrt{x}) \;=\; \int_0^\infty e^{-2\sqrt{x}\cosh(t)} t^n \cos\Big(\tau t + \frac{n\pi}{2}\Big)dt. \tag{2.4}$$

Which can be estimated as

$$\begin{aligned} |D_\tau^n K_{i\tau}(2\sqrt{x})| \;&\leq\; \int_0^\infty e^{-\sqrt{x}e^t}(1+t)^n dt \leq \int_0^\infty e^{-\sqrt{x}e^t} e^{tn} dt \\ &\leq\; \int_0^\infty u^{n-1}\, e^{-\sqrt{x}u} du = \frac{(n-1)!}{x^{\frac{n}{2}}} \end{aligned} \tag{2.5}$$

where $n \in \mathbb{N}$. Again from (2.4), we have

$$|D_\tau^n K_{i\tau}(2\sqrt{x})| \quad \leq \quad C \int_0^\infty e^{-2\sqrt{x}\cosh(t)} \cosh(t)dt = CK_1(2\sqrt{x}), \tag{2.6}$$

where $C > 0$ is constant depends upon $n \in \mathbb{N}$.
Next, we move on to the theory of translation and convolution operator associated to index KL-transform (1.7). In view of (1.7) and (1.9), we have

$$(\mathcal{K}K(x, y, \cdot))(\tau) = 2\tau K_{i\tau}(2\sqrt{x})K_{i\tau}(2\sqrt{y}). \tag{2.7}$$

Using (1.8) and (1.10), (2.7) can be written as

$$\exp\Big(- \Big(\frac{\mathrm{xy} + \mathrm{yz} + \mathrm{zx}}{\sqrt{\mathrm{xyz}}}\Big)\Big) = \frac{4}{\pi^2} \int_0^\infty \mathrm{K}_{\mathrm{i}\tau}(2\sqrt{\mathrm{x}})\mathrm{K}_{\mathrm{i}\tau}(2\sqrt{\mathrm{y}})\mathrm{K}_{\mathrm{i}\tau}(2\sqrt{\mathrm{z}})\tau \sinh(\pi\tau)\mathrm{d}\tau. \tag{2.8}$$

Moreover, we can define the translation operator associated to index KL-transform (1.7), as

$$(\mathcal{T}_x\varphi)(y) = \frac{1}{y} \int_0^\infty K(x, y, z)\varphi(z)zdz, \tag{2.9}$$

and the convolution operator corresponding to translation operator (2.9) as

$$(\varphi * \psi)(x) = \frac{1}{x} \int_0^\infty (\mathcal{T}_x\varphi)(y)\psi(y)ydy = \frac{1}{x} \int_0^\infty \int_0^\infty K(x, y, z)\varphi(z)\psi(y)\; z\; dy\; dz. \tag{2.10}$$

Theorem 2.1. *If translation and convolution operators associated to index KL-transform* (1.7) *are defined as above, then we have following operational relation*

$$(i) \quad (\mathcal{K}\mathcal{T}_x\varphi)(\tau) = K_{i\tau}(2\sqrt{x})(\mathcal{K}\varphi)(\tau), \tag{2.11}$$

$$(ii) \quad (\mathcal{K}(\varphi * \psi))(\tau) = \frac{1}{2\tau}(\mathcal{K}\varphi)(\tau)(\mathcal{K}\psi)(\tau). \tag{2.12}$$

Proof. (i) Using (1.7), (2.9), (1.9) and (1.10), (2.11) can be obtained easily.
(ii) Using (1.7), (2.10), (1.9) and (1.10), we have

$$\begin{aligned}
& (\mathcal{K}(\varphi * \psi))(\tau) \\
= \;& 2\tau \int_0^\infty \int_0^\infty \Big(\int_0^\infty K_{i\tau}(2\sqrt{x}) \exp\Big(- \Big(\frac{\mathrm{xy} + \mathrm{yz} + \mathrm{zx}}{\sqrt{\mathrm{xyz}}}\Big)\Big) \frac{\mathrm{dx}}{\mathrm{x}}\Big) \psi(\mathrm{y})\varphi(\mathrm{z})\; \mathrm{dz}\; \mathrm{dy} \\
= \;& 2\tau \int_0^\infty \int_0^\infty K_{i\tau}(2\sqrt{y})K_{i\tau}(2\sqrt{z})\psi(y)\varphi(z)\; dz\; dy \\
= \;& \frac{(\mathcal{K}\varphi)(\tau)}{2\tau}\, 2\tau \int_0^\infty K_{i\tau}(2\sqrt{y})\psi(y)dy = \frac{1}{2\tau}(\mathcal{K}\varphi)(\tau)(\mathcal{K}\psi)(\tau).
\end{aligned}$$

Hence, (2.12) is obtained. □

3 Estimates for translation and convolution operator

In this section, we obtain estimates of translation and convolution operators in Lebesgue space. Before moving to the estimation of convolution operator, we obtain estimate for the function $K(x, y, z)$. For all $x,\; y,\; z \in \mathbb{R}_+$, we can write

$$\sqrt{\frac{xy}{z}} + \sqrt{\frac{yz}{x}} + \sqrt{\frac{zx}{y}} \quad > \quad \sqrt{x}\Big(\sqrt{\frac{y}{z}} + \sqrt{\frac{z}{y}}\Big) \geq 2\sqrt{x} \; > \; \sqrt{x}.$$

Thus

$$exp\Big(-\Big(\sqrt{\frac{xy}{z}}+\sqrt{\frac{yz}{x}}+\sqrt{\frac{zx}{y}}\Big)\Big) \leq e^{-\sqrt{x}}. \tag{3.1}$$

Similarly, we can obtain

$$exp\Big(-\Big(\sqrt{\frac{xy}{z}}+\sqrt{\frac{yz}{x}}+\sqrt{\frac{zx}{y}}\Big)\Big) \quad \leq \quad e^{-\sqrt{y}} \text{ or } e^{-\sqrt{z}}. \tag{3.2}$$

Using (3.1), (3.2) and (1.4), we get

$$|K(x,y,z)| \quad \leq \quad \frac{e^{-\sqrt{x}}}{z} \text{ or } \frac{e^{-\sqrt{y}}}{z} \text{ or } \frac{e^{-\sqrt{z}}}{z}. \tag{3.3}$$

Theorem 3.1. *If the translation and convolution operators are defined as* (2.9) *and* (2.10) *respectively, then*

$$(i) \quad \|\mathcal{T}_x f\|_{L^1(\mathbb{R}_+;\, ydy)} \;\leq\; C\, \|f\|_{L^1(\mathbb{R}_+;\, dz)}, \tag{3.4}$$

$$(ii) \quad \|f*g\|_{L^1(\mathbb{R}_+;\, xdx)} \;\leq\; C\|f\|_{L^1(\mathbb{R}_+;\, dz)}\|g\|_{L^1(\mathbb{R}_+;\, dy)}, \tag{3.5}$$

where C is some positive constant.

Proof. (i) Using (2.9) and (3.3), we get

$$|\mathcal{T}_x f(y)| \quad \leq \quad \frac{e^{-\sqrt{y}}}{y}\int_0^\infty |f(z)|\, dz.$$

Hence

$$\|\mathcal{T}_x f\|_{L^1(\mathbb{R}_+;\, ydy)} \quad \leq \quad C\, \|f\|_{L^1(\mathbb{R}_+;\, dz)},$$

where C is some positive constant.
(ii) Using (2.10) and (3.3), we get

$$|(f*g)(x)| \quad \leq \quad \frac{e^{-\sqrt{x}}}{x}\int_0^\infty\int_0^\infty f(z)g(y)\, dy\, dz.$$

Hence

$$\|f*g\|_{L^1(\mathbb{R}_+;\, xdx)} \quad \leq \quad C\|f\|_{L^1(\mathbb{R}_+;\, dz)}\|g\|_{L^1(\mathbb{R}_+;\, dy)},$$

where C is some positive constant. □

Theorem 3.2. *If the translation and convolution operators are defined as* (2.9) *and* (2.10) *respectively, then*

$$(i) \quad \|(\cdot)\mathcal{T}_x f\|_{L^p(\mathbb{R}_+;\, dy)} \;\leq\; C\, \|f\|_{L^p(\mathbb{R}_+;\, dz)}, \tag{3.6}$$

$$(ii) \quad \|(\cdot)(f*g)\|_{L^p(\mathbb{R}_+;\, dx)} \;\leq\; C\|f\|_{L^p(\mathbb{R}_+;\, dz)}\|g\|_{L^1(\mathbb{R}_+;\, dy)}, \tag{3.7}$$

where C is some positive constant and $1 \;\leq\; p < \infty$.

Proof. (i) Using (2.9) and Hölder;s inequality, we have

$$|\mathcal{T}_x f(y)| \quad \leq \quad \frac{1}{y}\Big(\int_0^\infty |K(x,y,z)|\, |f(z)|^p z\, dz\Big)^{\frac{1}{p}}\Big(\int_0^\infty |K(x,y,z)|\, z\, dz\Big)^{\frac{1}{q}}.$$

Further, using (3.3), we get

$$|y\mathcal{T}_x f(y)|^p \leq e^{-\sqrt{y}}\left(\int_0^\infty |f(z)|^p\, dz\right)\left(\int_0^\infty e^{-\sqrt{z}}\, dz\right)^{\frac{p}{q}}.$$

Thus

$$\|(\cdot)\mathcal{T}_x f\|_{L^p(\mathbb{R}_+;\, dy)} \leq C\, \|f\|_{L^p(\mathbb{R}_+;\, dz)},$$

where C is some positive constant.
(ii) Using (2.10) and Hölder's inequality, we have

$$|(f*g)(x)| \leq \frac{1}{x}\left(\int_0^\infty\int_0^\infty |K(x,y,z)|\, |f(z)|^p |g(y)|\, z\, dy\, dz\right)^{\frac{1}{p}}$$
$$\times \left(\int_0^\infty\int_0^\infty |K(x,y,z)|\, |g(y)|\, z\, dy\, dz\right)^{\frac{1}{q}}.$$

Further, using (3.3), we get

$$\int_0^\infty |x(f*g)(x)|^p dx \leq \left(\int_0^\infty e^{-\sqrt{x}} dx\right)\|f\|^p_{L^p(\mathbb{R}_+;\, dz)}\left(\int_0^\infty |g(y)|\, dy\right)$$
$$\times \left(\int_0^\infty e^{-\sqrt{z}}\, dz\right)^{\frac{p}{q}}\left(\int_0^\infty |g(y)|\, dy\right)^{\frac{p}{q}}.$$

Hence

$$\|(\cdot)(f*g)\|_{L^p(\mathbb{R}_+;\, dx)} \leq C\|f\|_{L^p(\mathbb{R}_+;\, dz)}\|g\|_{L^1(\mathbb{R}_+;\, dy)},$$

where C is some positive constant. □

4 Continuity of index KL-transform

In this section, we discuss the continuity of index KL-transform in certain suitable function space.
Let us consider the function space $\mathcal{G}^{m,n}$

Definition 4.1. The function space $\mathcal{G}^{m,n}$, $m \in [0,\infty)$, $n \in \mathbb{N}_0$ consists of all complex-valued infinitely differentiable functions f, defined on R+ such that

$$\lambda_{m,n}(f) = \sup_{x\in\mathbb{R}_+} |e^{-mx} D_x^n f(x)| < \infty. \tag{4.1}$$

Theorem 4.2. *The index KL-transform* (1.6)*, is continuous linear mapping from* $\mathcal{G}^{0,0}$ *into* $\mathcal{G}^{s,n}$*, where* $s \in (0,\infty)$*,* $n \in \mathbb{N}_0$.

Proof. Using (1.6), Leibnitz rule of differentiation and (2.6), we have

$$|e^{-s\tau} D_\tau^n(\mathcal{K}f)(\tau)| \leq 2C\, e^{-s\tau}\sum_{r=0}^n \binom{n}{r} D_\tau^r \tau \int_0^\infty K_1(2\sqrt{x})|f(x)|dx.$$

Thus, in view of (4.1), we have

$$\lambda_{s,n}(\mathcal{K}f) \leq 2c\, \lambda_{0,0}(f)\sum_{r=0}^n \binom{n}{r} \sup_{\tau\in\mathbb{R}_+} e^{-s\tau} D_\tau^r \tau \int_0^\infty K_1(2\sqrt{x})dx.$$

Clearly the involved integral is finite as supremum is finite for $s \in (0, \infty),\ n \in \mathbb{N}_0$, so from above we can write

$$\lambda_{s,n}(\mathcal{K}f) \quad \leq \quad C'\lambda_{0,0}(f) < \infty,$$

where C' is some positive constant. Hence, the theorem is proved. □

We consider another function space, similar to above where operator D_x is replaced by the operator $\mathcal{A}_x$ as (1.4).

Definition 4.3. The function space $\mathcal{H}^{m,n},\ m \in [0, \infty),\ n \in \mathbb{N}_0$ consists of all complex-valued infinitely differentiable functions f, defined on R+ such that

$$\Gamma_{m,n}(f) = \sup_{x \in \mathbb{R}_+} |e^{-mx}\mathcal{A}_x^n f(x)| \ < \ \infty. \tag{4.2}$$

Theorem 4.4. *The differential operator $\mathcal{A}_x$ is continuous linear mapping from $\mathcal{H}^{m,n}$ onto itself.*

Proof. Using (4.2), proof is straightforward. □

Theorem 4.5. *The index KL-transform (1.6), is continuous linear mapping from $\mathcal{H}^{0,0}$ into $\mathcal{H}^{s,n}$, where $s \in (0, \infty),\ n \in \mathbb{N}_0$..*

Proof. Proof can be done easily by using (1.6), and proceeding as proof of above theorem. □

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