Palestine Journal of Mathematics

Vol. 11(Special Issue II)(2022) , 1–12

Qualitative dynamics of quadratic systems exhibiting reducible invariant algebraic curve of degree 3

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Communicated by Mohamed Kharrat

MSC 2010 Classifications: Primary 34C15, Secondary 34C25.

Keywords and phrases: Quadratic differential system, Poincaré disc, invariant algebraic curve, reducible cubic curve, global phase portraits.

Abstract. We intend in this article to classify the global phase portraits of quadratic polynomial differential systems exhibiting reducible invariant algebraic curve of degree three, by investigating their global phase portraits in the Poincaré disc. We realize that these systems produce 13 topologically different phase portraits.

1 Introduction

Let $\mathbb{R}[x, y]$ be the ring of real polynomials in the variables x and y. A quadratic polynomial differential system is a system of the form

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \tag{1.1}$$

where P and Q are real polynomials form $\mathbb{R}[x, y]$ and $\max\{\deg(P), \deg(Q)\} = 2$. The variables $\dot{x}$ and $\dot{y}$ are obviously the derivatives os x and y with respect to the time t. For such system, we can always associate the quadratic vector field

$$\mathcal{X} = P(x, y)\partial/\partial x + Q(x, y)\partial/\partial y.$$

If system (1.1) has an algebraic trajectory curve, which is defined by a zero set of a polynomial $H(x, y) = 0$. Then it is clear that the derivative of H with respect to the time will not change along the curve $H = 0$, and by the Hilbert's Nullstellensatz we have

$$\frac{dH}{dt} = \frac{\partial H}{\partial x}P + \frac{\partial H}{\partial y}Q = KH,$$

where K is a polynomial in x and y of degree at most 1, called the cofactor of the invariant algebraic curve $H(x, y) = 0$. For more details on the invariant algebraic curves of a polynomial differential system see Chapter 8 of [7].

The quadratic differential systems gained a big notoriety and has been extensively explored using different mathematical tools and methods, for more details see the references cited in the books of Ye [13] and Reyn [12].

In [5] Benterki and Llibre studied the global phase portraits of 14 quadratic polynomial differential systems having 14 classical quartic algebraic curves as invariant ones, which are formed by orbits of the quadratic polynomial differential systems, and they obtained 28 global phase portraits topologically non equivalent.

In [3] Belfar and Benterki classified the global phase portraits of six quadratic polynomial differential systems, exhibiting as unvariant algebraic curves six well-known algebraic curves of degree six. The same authors in [4] classified the dynamics of five quadratic differential systems exhibiting five known different cubic invariant algebraic curves, and they realized that these systems produced 29 topologically different phase portraits.

In this work, we aim to characterize the global phase portraits in the Poincaré disc of a quadratic systems having reducible invariant cubic algebraic curve.

2 Statement of the main results

Our first main result is the following.

Theorem 2.1. *The algebraic curve of degree three given by:* $H(x, y) = 0$ *with* $H(x, y) = (y - k)(x^2 + y^2 - 1)$ *where* $k \neq 0$*, is an invariant algebraic curve with associated cofactor* $K(x, y) = ax$ *of the quadratic differential systems:*

$$\begin{aligned} \dot{x} &= \frac{1}{2}(a-1)x^2 + \frac{1}{2}(a-3)y^2 + ky + \frac{1-a}{2}, \\ \dot{y} &= x(y-k). \end{aligned} \tag{2.1}$$

Proof. It is immediate that the function H on the orbits of systems (2.1) satisfy

$$\frac{dH}{dt} = \dot{x}\frac{\partial H}{\partial x} + \dot{y}\frac{\partial H}{\partial y} = KH.$$

The following theorem characterize the topological classifcation of all the phase portraits of planar polynomial diferential systems of degree 2 having the reducible invariant cubic curve $H(x, y) = (y - k)(x^2 + y^2 - 1) = 0$ in the Poincaré disc. For a defnition of singular points and Poincaré disc and for a defnition of a topological equivalent phase portraits of a polynomial diferential system in the Poincaré disc see sections 2.

Theorem 2.2. *The global phase portraits of the quadratic systems* (2.1) *are topologically equivalent to the phase portrait*

1 *for* $k \in (0, 1)$ *and* $a \in (1, c_1) \cup (c_2, 3) \cup (3, \infty)$*, where* $c_1 = 2 - \sqrt{1 - k^2}$ *and* $c_2 = 2 + \sqrt{1 - k^2}$*;*

2 *for* $k \in (0, 1)$ *and* $a \in (-\infty, 1)$*;*

3 *for* $k \in (0, 1)$ *and* $a \in (c_1, c_2)$*;*

4 *for* $k \in (0, 1)$ *and* $a = c_1$*;*

5 *for* $k \in (0, 1)$ *and* $a = c_2$*;*

6 *for* $k \in (0, \infty)$ *and* $a = 1$*;*

7 *for* $k \in (0, 1)$ *and* $a = 3$*;*

8 *for* $k \in (1, \infty)$ *and* $a \in (-\infty, 1) \cup (3, \infty)$*;*

9 *for* $k \in (1, \infty)$ *and* $a \in (1, 3)$ *;*

10 *for* $k \in (1, \infty)$ *and* $a = 3$*;*

11 *for* $k = 1$ *and* $a = 2$*;*

12 *for* $k = 1$ *and* $a = 3$*;*

13 *for* $k = 1$ *and* $a \in (-\infty, 1) \cup (1, 2) \cup (2, 3)$*;*

14 *for* $k = 1$ *and* $a \in (3, \infty)$.

3 Preliminaries and basic results

In this section we expose the main results and concepts which are necessary for proving our results.

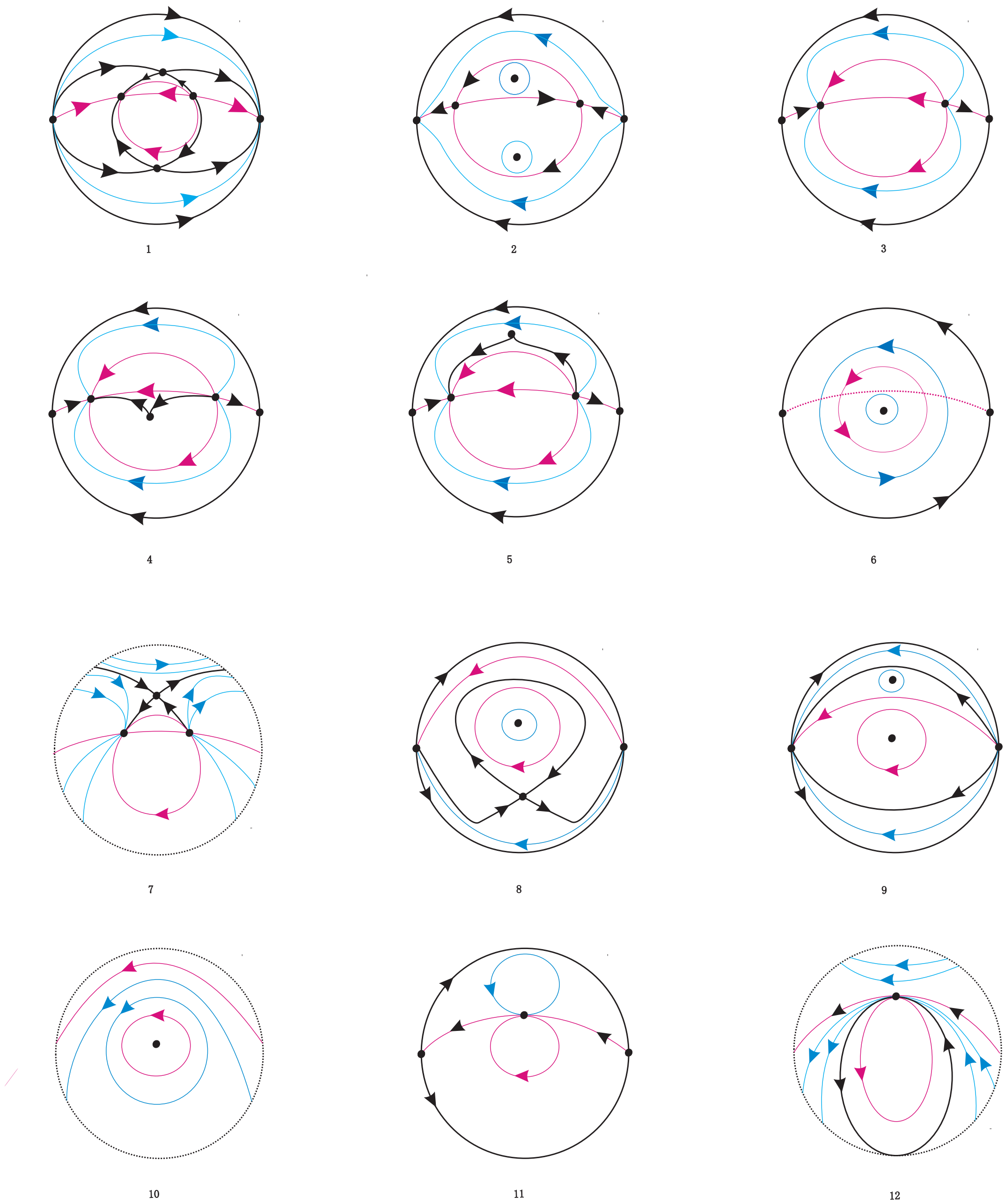

Figure 1. Phase portraits in the Poincaré disc of the systems (2.1). The invariant algebraic curves of degree 3 are drawn in red color. An orbit inside a canonical region is drawn in blue except if it is contained in the invariant algebraic curve. The separatrices are drawn in black except if the separatrix is contained in the invariant algebraic curve then it is of red color but its arrow is black in order to indicate that is a separatrix.

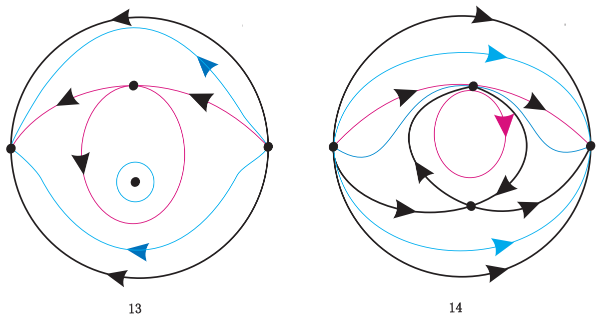

Figure 2. Continuation of Figure 1.

3.1 Poincaré compactification

In this subsection, we present some basic results that we need to study the behavior of the trajectories of a planar differential systems near infinity. Let $X(x,y) = (\frac{1}{2}(a-1)x^2 + \frac{1}{2}(a-3)y^2 + ky + \frac{1-a}{2}, x(y-k))$ represent a vector field of systems (2.1) which we are going to study their phase portraits, then for doing this we use the so called a Poincaré compactification. We consider the Poincaré sphere $^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 1\}$, and we define the central projection $f : T^2_{(0,0,1)} \longrightarrow^2$ (with $T^2_{(0,0,1)}$ the tangent space of ^{2} at the point $(0,0,1)$, such that for each point $q \in T^2_{(0,0,1)}$, $T^2_{(0,0,1)}(q)$ associates the two intersection points of the straight line which connects the point q and $(0,0)$. The equator $^1 = \{(x,y,z) \in^2 : z = 0\}$ represent the infinity points of $\mathbb{R}^2$. In summary we get a vector field $\mathcal{X}'$ defined in $^2 \backslash ^1$, which is formed by to symmetric copies of $\mathcal{X}$, and we prolong it to a vector field $p(\mathcal{X})$ on ^{2}. By studying the dynamics of $p(\mathcal{X})$ near ^{1} we get the dynamics of $\mathcal{X}$ at infinity. We need to do the calculations on the Poincaré sphere near the local charts $U_i = \{Y \in^2 : y_i > 0\}$, and $V_i = \{Y \in^2 : y_i < 0\}$ for $i = 1,2,3$; with the associated diffeomorphisms $F_i : U_i \longrightarrow \mathbb{R}^2$ and $G_i : V_i \longrightarrow \mathbb{R}^2$ for $i = 1,2,3$. After a rescaling in the independent variable in the local chart (U_1, F_1) the expression for $p(\mathcal{X})$ is

$$\dot{u} = v^n \left[-uP\left(\frac{1}{v}, \frac{u}{v}\right) + Q\left(\frac{1}{v}, \frac{u}{v}\right) \right], \qquad \dot{v} = -v^{n+1} P\left(\frac{1}{v}, \frac{u}{v}\right);$$

in the local chart (U_2, F_2) the expression for $p(\mathcal{X})$ is

$$\dot{u} = v^n \left[P\left(\frac{u}{v}, \frac{1}{v}\right) - uQ\left(\frac{u}{v}, \frac{1}{v}\right) \right], \qquad \dot{v} = -v^{n+1} Q\left(\frac{u}{v}, \frac{1}{v}\right);$$

and for the local chart (U_3, F_3) the expression for $p(\mathcal{X})$ is

$$\dot{u} = P(u,v), \qquad \dot{v} = Q(u,v).$$

Due to the fact that the singular points at infinity appear in pairs diametrally opposite, then for studying the local phase portrait of a singular point at infinity, we have to study the singular points $(u_0, 0)$ at the local chart U_1, and the origin of the chart U_2.

For more details on the Poincaré compactification see Chapter 5 of [7].

3.2 Phase portraits on the Poincaré disc

In this subsection we are going to see how to characterize the global phase portraits in the Poincaré disc of all the differential systems (2.1).

A *separatrix* of $p(\mathcal{X})$ is an orbit which is either a singular point, or a limit cycle, or a trajectory which lies in the boundary of an hyperbolic sector at a singular point. Neumann [10] proved that the set formed by all separatrices of $p(\mathcal{X})$; denoted by $S(p(\mathcal{X}))$ is closed.

The open connected components of $\mathbb{D}^2 \setminus S(p(\mathcal{X}))$ are called *canonical regions* of $p(\mathcal{X})$: We define a *separatrix configuration* as a union of $S(p(\mathcal{X}))$ plus one solution chosen from each canonical region. Two separatrix configurations $S(p(\mathcal{X}))$ and $S(p(\mathcal{Y}))$ are said to be *topologically equivalent* if there is an orientation preserving or reversing homeomorphism which maps the trajectories of $S(p(\mathcal{X}))$ into the trajectories of $S(p(\mathcal{Y}))$.

The following result is due to Markus [9], Neumann [10] and Peixoto [11].

Theorem 3.1. *The phase portraits in the Poincaré disc of the two compactified polynomial differential systems $p(\mathcal{X})$ and $p(\mathcal{Y})$ are topologically equivalent if and only if their separatrix configurations $S(p(\mathcal{X}))$ and $S(p(\mathcal{Y}))$ are topologically equivalent.*

According to this theorem and in the phase portraits in the Poincaré disc of Figures 1 and 2 we plot at least one orbit in each canonical region.

Remark 3.2. Systems (2.1) are invariant under the change $(x, y, t, a, k) \longrightarrow (-x, -y, t, a, -k)$, then we only need to study them for $k > 0$.

To characterize the phase portraits of systems (2.1) we have to:

First of all we have to describe and study the finite singular points and their local phase portraits. Then we repeat the same process for the infnite singularities.

3.3 The finite singular points

By considering the symmetry according to Remark (3.2), the finite singular points of systems (2.1) are given by:

Proposition 3.3. *The following statements hold for the quadratic systems (2.1).*

(I) Assume $k \in (0, 1)$

- (i) *If $a \in (1, c_1) \cup (c_2, 3) \cup (3, \infty)$ where $c_1 = 2 - \sqrt{1-k^2}$ and $c_2 = 2 + \sqrt{1-k^2}$, then systems (2.1) have four singularities, an hyperbolic stable node at $p_1 = (-\sqrt{1-k^2}, k)$, an hyperbolic unstable node at $p_2 = (\sqrt{1-k^2}, k)$; the third singularity at $p_3 = \left(0, \dfrac{k + \sqrt{a^2 - 4a + k^2 + 3}}{3-a}\right)$ which is an hyperbolic saddle if $a \in (1, c_1) \cup (3, \infty)$ and a center if $a \in (c_2, 3)$; the fourth singular point at $p_4 = \left(0, \dfrac{k - \sqrt{a^2 - 4a + k^2 + 3}}{3-a}\right)$ which is an hyperbolic saddle if $a \in (c_2, 3) \cup (3, \infty)$ and a center if $(1, c_1)$.*
- (ii) *If $a \in (-\infty, 1)$ systems (2.1) have four singularities, two hyperbolic saddles at p_1 and p_2, and two centers at p_3 and p_4.*
- (iii) *If $a \in (c_1, c_2)$ systems (2.1) have two hyperbolic singularities, a stable node at p_1 and an unstable node at p_2.*
- (iv) *If $a = c_1$ systems (2.1) have three singularities, an hyperbolic stable node at p_1, an hyperbolic unstable node at p_2 and a nilpotent singularity at $p_3 = \left(0, \dfrac{1 - \sqrt{1-k^2}}{k}\right)$, where its local phase portrait formed by two hyperbolic sectors.*
- (v) *If $a = c_2$ systems (2.1) have three singularities, an hyperbolic stable node at p_1, an hyperbolic unstable node at p_2 and a nilpotent singularity at $p_4 = \left(0, \dfrac{1 + \sqrt{1-k^2}}{k}\right)$, where its local phase portrait formed by two hyperbolic sectors.*
- (vi) *If $a = 1$ systems (2.1) have $y = k$ as a line of singularities, and by doing the change of variables $(y - k)dt = ds$ we know that the system has a center at the origin.*
- (vii) *If $a = 3$ systems (2.1) have three hyperbolic singularities, a stable node at p_1, an unstable node at p_2 and a saddle at $p_4 = \left(0, \dfrac{1}{k}\right)$.*

(II) Assume $k \in (1, \infty)$

- (i) *If $a \in (-\infty, 1) \cup (3, \infty)$ systems (2.1) have two singularities, a hyperbolic saddle at p_3 and a center at p_4.*
- (ii) *If $a \in (1, 3)$ they have two centers at p_3 and p_4.*
- (iii) *If $a = 1$ they have $y = k$ as a line of singularities, and by doing the change of variables $(y - k)dt = ds$ we know that the system has a center at the origin.*
- (iv) *If $a = 3$ they have one singularity at p_4 which is a center.*

(III) Assume $k = 1$

- (i) *If $a = 1$ system (2.1) has $y = 1$ as a line of singularities, and by doing the change of variables $(y - 1)dt = ds$ we know that it has a center at the origin.*
- (ii) *If $a = 2$ the system has one finite singularity at $p_3 = (0, 1)$ which is a linearly zero, and its local phase portrait formed by two elliptic sectors.*
- (iii) *If $a = 3$ it has one finite singularity at p_3 which is a nilpotent, and its local phase portrait formed by two parabolic and one hyperbolic sectors.*
- (iv) *If $a \in (-\infty, 1) \cup (1, 2) \cup (2, 3) \cup (3, \infty)$ systems (2.1) have two singularities; p_3 which is nilpotent, and its local phase portrait formed by two parabolic, one elliptic and one hyperbolic sectors if $a \in (3, \infty)$ and four hyperbolic sectors if $a \in (-\infty, 1) \cup (1, 2) \cup (2, 3)$; $p_4 = \left(0, \dfrac{1-a}{a-3}\right)$ which is an hyperbolic saddle if $a \in (3, \infty)$ and a center if $a \in (-\infty, 1) \cup (1, 2) \cup (2, 3)$.*

Proof.

Proof of statement (I). *If $a \in (1, c_1) \cup (c_2, 3) \cup (3, \infty)$ the differential systems (2.1) have four singularities, an hyperbolic stable node at p_1 which has the eigenvalues $\lambda_1 = -(a-1)\sqrt{1-k^2}$ and $\lambda_2 = -\sqrt{1-k^2}$, and an hyperbolic unstable node at p_2 with its corresponding eigenvalues $\lambda_1 = \sqrt{1-k^2}$ and $\lambda_2 = (a-1)\sqrt{1-k^2}$, and the third singular point p_3 has $\lambda_{1,2} = \mp iB$ such that $B = \sqrt{-S((a-2)k+S)/(a-3)}$ and $S = \sqrt{(a-4)a+k^2+3}$, then $\lambda_1.\lambda_2 = -\dfrac{S\left((a-2)k+S\right)}{a-3}$. According to the sign of the parameter a we know that p_3 is an hyperbolic saddle if $a \in (1, c_1) \cup (3, \infty)$ and a center if $a \in (c_2, 3)$.*

The fourth singularity p_4 has $\lambda_{1,2} = \mp iB$ such that $B = \sqrt{S((a-2)k-S)/(a-3)}$ and $S = \sqrt{(a-4)a+k^2+3}$, then $\lambda_1.\lambda_2 = \dfrac{S\left((a-2)k-S\right)}{a-3}$. According to the sign of the parameter a we know that p_4 is an hyperbolic sadlle if $a \in (c_2, 3) \cup (3, \infty)$ and a center if $a \in (1, c_1)$. Then the statement (i) holds.

If $a \in (-\infty, 1)$ the differential systems (2.1) have four singularities, an hyperbolic stable node at p_1 which has the eigenvalues $\lambda_1 = -(a-1)\sqrt{1-k^2}$ and $\lambda_2 = -\sqrt{1-k^2}$, and an hyperbolic unstable node at p_2 with its corresponding eigenvalues $\lambda_1 = \sqrt{1-k^2}$ and $\lambda_2 = (a-1)\sqrt{1-k^2}$, and the third singular point p_3 has the eigenvalues $\lambda_{1,2} = \mp iB$ such that $B = \sqrt{-S((a-2)k+S)/(a-3)}$ and $S = \sqrt{(a-4)a+k^2+3}$. These eigenvalues are purely imaginary such this equilibrum point is either a focus or a center, but due to the fact that system (2.1) is symetric with respect to $(xx^{'})$ axes, P_3 is a center, and a fourth singular point p_4 with eigenvalues $\lambda_{1,2} = \mp iB$ such that $B = \sqrt{S((a-2)k-S)/(a-3)}$ and $S = \sqrt{(a-4)a+k^2+3}$. This eigenvalues are imaginary purely which means that the singularity is either a focus or a center, but due to the fact that systems (2.1) are symetric with respect to the x-axes, we know that P_4 is a center. Then the statement (ii) holds.

If $a \in (c_1, c_2)$, then p_1 are an hyperbolic stable node with eigenvalues $\lambda_1 = -(a-1)\sqrt{1-k^2}$ and $\lambda_2 = -\sqrt{1-k^2}$ and p_2 is an unstable node with eigenvalues $\lambda_1 = \sqrt{1-k^2}$ and $\lambda_2 = (a-1)\sqrt{1-k^2}$. Then the statement (iii) holds.

If $a = c_1$ where $c_1 = 2 - \sqrt{1-k^2}$, the differential systems (2.1) become

$$\begin{aligned} \dot{x} &= \frac{1}{2}\left(1-\sqrt{1-k^2}\right)x^2 + \frac{1}{2}\left(-\sqrt{1-k^2}-1\right)y^2 + ky + \frac{1}{2}\left(\sqrt{1-k^2}-1\right), \\ \dot{y} &= x(y-k). \end{aligned} \tag{3.1}$$

These systems have three singularities p_1, p_2 and p_3, with $p_3 = \left(0, \dfrac{1-\sqrt{1-k^2}}{k}\right)$.

At p_1 we have the eigenvalues $\lambda_1 = \sqrt{1-k^2}\left(\sqrt{1-k^2}-1\right)$ and $\lambda_2 - \sqrt{1-k^2}$. So it's an hyperbolic stable node. The singularity p_2 is an hyperbolic unstable node with eigenvalues $\lambda_1 = \sqrt{1-k^2}$ and $\lambda_2 = -\sqrt{1-k^2}\left(\sqrt{1-k^2}-1\right)$. The third singularity p_3 is a nilpotent singular point with eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 0$. By applying Theorem 2.15 of [7] we know that p_3 has two hyperbolic sectors. Then the statement (iv) holds.

If $a = c_2$ where $c_2 = 2 + \sqrt{1-k^2}$ the differential systems (2.1) become

$$\begin{aligned} \dot{x} &= \frac{1}{2}\left(1+\sqrt{1-k^2}\right)x^2 + \frac{1}{2}\left(\sqrt{1-k^2}-1\right)y^2 + ky + \frac{1}{2}\left(-\sqrt{1-k^2}-1\right), \\ \dot{y} &= x(y-k). \end{aligned} \tag{3.2}$$

These systems have three singularities p_1, p_2 and p_4, with $p_4 = \left(0, \dfrac{1+\sqrt{1-k^2}}{k}\right)$.

The singularity p_1 has the eigenvalues $\lambda_1 = -\sqrt{1-k^2}$ and $\lambda_2 = -\sqrt{1-k^2}\left(\sqrt{1-k^2}+1\right)$. So it's an hyperbolic stable node. The singularity p_2 is an unstable node with eigenvalues $\lambda_1 = \sqrt{1-k^2}\left(\sqrt{1-k^2}+1\right)$ and $\lambda_2 = \sqrt{1-k^2}$. The third singularity p_4 is a nilpotent singular point with eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 0$. By applying Theorem 3.5 of [7] we know that p_4 has two hyperbolic sectors. Then the statement (v) holds.

If $a = 1$ the differential systems (2.1) become

$$\dot{x} = -y(y-k), \qquad \dot{y} = x(y-k). \tag{3.3}$$

They have a line of singularities $y = k$. We take the change of variables $ds = (y-k)dt$, we get the following system

$$\dot{x} = -y, \quad \dot{y} = x. \tag{3.4}$$

which has a center at the origin with its corresponding eigenvalues i and $-i$. Then the statement (vi) holds.

If $a = 3$ the differential systems (2.1) become

$$\dot{x} = x^2 + ky - 1, \qquad \dot{y} = x(y-k). \tag{3.5}$$

They have three singularities p_1, p_2 and p_4 where $p_4 = \left(0, \dfrac{1}{k}\right)$.

The singularities p_1 and p_2 have the eigenvalues $\lambda_1 = -2\sqrt{1-k^2}$ and $\lambda_2 = -\sqrt{1-k^2}$, and $\lambda_1\sqrt{1-k^2}$ and $\lambda_2 = 2\sqrt{1-k^2}$, respectively. So p_1 is an hyperbolic stable node and p_2 is an hyperbolic unstable node. The third singularity p_4 is a saddle with eigenvalues $\lambda_1 = -\sqrt{1-k^2}$ and $\lambda_2 = \sqrt{1-k^2}$. Then the statement (vii) holds.

Proof of statement (II). If $a \in (-\infty, 1) \cup (1, 3) \cup (3, \infty)$ the systems have two singularities; p_3 with eigenvalues $\lambda_{1,2} = \mp iB$ such that $B = \sqrt{-S((a-2)k+S)/(a-3)}$ and $S = \sqrt{(a-4)a+k^2+3}$, then $\lambda_1.\lambda_2 = -\dfrac{S\left((a-2)k+S\right)}{a-3}$ which means that p_3 is an hyperbolic saddle if $a \in (-\infty, 1) \cup (3, \infty)$ and a center if $a \in (1, 3)$, and the second singular point p_4 has the eigenvalues $\lambda_{1,2} = \mp iB$ such that $B = \sqrt{S((a-2)k-S)/(a-3)}$ and $S = \sqrt{(a-4)a+k^2+3}$. These eigenvalues are purely imaginary, then this equilibrum point is either a focus or a center, but due to the fact that systems (2.1) are symetric with respect to the x–axes we know that P_4 is a center. Then two statements (i) and (ii) hold.

If $a = 1$ the differential systems (2.1) become

$$\dot{x} = -y(y-k), \qquad \dot{y} = x(y-k). \tag{3.6}$$

These systems have a line of singularities $y = k$. By performing the change of variables $ds = (y-k)dt$, we get the following system

$$\dot{x} = -y, \qquad \dot{y} = x. \tag{3.7}$$

This system has a center at the origin with its corresponding eigenvalues $\lambda_1 = i$ and $\lambda_1 = -i$. Then the statement (iii) holds.

If $a = 3$ the differential systems (2.1) become

$$\dot{x} = x^2 + ky - 1, \qquad \dot{y} = x(y-k). \tag{3.8}$$

These systems have one singularity at p_4 which is a center with eigenvalues $\lambda_1 = -i\sqrt{1-k^2}$ and $\lambda_2 = i\sqrt{1-k^2}$. Then the statement (iv) holds.

Proof of statement (III). If $a = 1$ systems (2.1) become

$$\dot{x} = -y(y-1), \qquad \dot{y} = x(y-1). \tag{3.9}$$

These systems have a line of singularities $y = 1$. We take the change of variable $ds = (y-1)dt$, we get the following system

$$\dot{x} = -y, \quad \dot{y} = x, \tag{3.10}$$

which has a center at the origin and its corresponding eigenvalues are $\lambda_1 = i$ and $\lambda_1 = -i$. Then the statement (i) holds.

If $a = 2$ the differential systems (2.1) become

$$\dot{x} = \frac{x^2}{2} - \frac{y^2}{2} + y - \frac{1}{2}, \qquad \dot{y} = x(y-1). \tag{3.11}$$

These systems have a linearly zero singular point at $p_3 = (0,1)$. In order to know the nature of this singularity. First, we put this point at the origin of coordinates by performing the translation $x = x_1$, $y = y_1 + 1$, and we get

$$\dot{x_1} = \frac{1}{2}(x_1 - y_1)(x_1 + y_1), \qquad \dot{y_1} = x_1 y_1. \tag{3.12}$$

Second, we need to do a blow-up $y_1 = zx_1$ for describing its local phase portrait. After eliminating the common factor x_1 of $\dot{x}_1$ and $\dot{z}$, by doing the rescaling of the independent variable $ds = x_1 dt$, and we obtain the system

$$\dot{x_1} = \frac{1}{2}(1-z)(1+z), \qquad \dot{z} = \frac{1}{2}z(1+z^2). \tag{3.13}$$

This system has no singularity for $x_1 = 0$. Going back through the two changes of variables $y_1 = zx_1$ and $x_1 dt = ds$ and by taking into account the direction of the flow of the system on the axes of coordinates, we conclud that the local phase portrait of the origin consists of two elliptic sectors. Then the statement (ii) holds.

If $a = 3$ systems (2.1) become

$$\dot{x} = x^2 + y - 1, \qquad \dot{y} = x(y-1). \tag{3.14}$$

This system has one nilpotent singularity at p_3. By using Theorem 3.5 of [7] we obtain that its local phase portrait consists of two parabolic and one hyperbolic sectors. Then the statement (iii) holds.

If $a \in (-\infty, 1) \cup (1,2) \cup (2,3) \cup (3,\infty)$ systems (2.1) have two singularities; a nilpotent singularity at p_3 and an hyperbolic singularity at p_4 with eigenvalues $-\sqrt{2}\sqrt{\dfrac{(a-2)^2}{a-3}}$ and $\sqrt{2}\sqrt{\dfrac{(a-2)^2}{a-3}}$, then it is a saddle if $a \in (3,\infty)$ and a center if $a \in (-\infty,1) \cup (1,2) \cup (2,3)$.

By using Theorem 3.5 of [7] we obtain that the local phase portrait of p_3 consists of two parabolic, one elliptic and one hyperbolic sectors if $a \in (3,\infty)$ and four hyperbolic sectors if $a \in (-\infty,1) \cup (1,2)$. Then the statement (iv) holds.

3.4 Infinite singular points

We aim to survey the behaviour of the local phase portraits of systems (2.1) *at their infinite singular points. To investigate the infinite singular points in the Poincaré disc we analyse the vector field at infinity.*

Proposition 3.4. *The local phase portraits at the infinite singular points of systems* (2.1) *in the local chart* U_1 *consists of*

(i) One singular point at $q_1 = (0,0)$ *which is an hyperbolic stable node if* $a \in (-\infty, 1)$ *and* $k \in (0,\infty)$*, an hyperbolic unstable node if* $a \in (3,\infty)$ *and* $k \in (0,\infty)$*, a hyperbolic saddle if* $a \in (1,3)$ *and* $k \in (0,\infty)$*, and a semi-hyperbolic saddle-node if* $a = 1$ *and* $k \in (0,\infty)$*;*

(ii) a line of singularities if $a = 3$ *and* $k \in (0,\infty)$*.*

The origin of the local chart U_2 *is not a singularity for all* $a \in \mathbb{R}$ *and* $k \in (0,\infty)$*.*

Proof. *The expression of systems* (2.1) *in the local chart* U_1 *is given by*

$$\begin{aligned} \dot{u} &= \frac{1}{2}\left(-(a-3)u^3 + u\left(a\left(v^2-1\right) - v^2 + 3\right) - 2ku^2v - 2kv\right) + a(1+v))), \\ \dot{v} &= -\frac{1}{2}v\left(-a\left(u^2+1\right) + (a-1)v^2 - 2kuv + 3u^2 + 1\right). \end{aligned} \tag{3.15}$$

Any arbitrary infinite singular point of differential systems (3.15) *take the forme* $(u_0, 0)$*.*

If $a \in (-\infty,3) \cup (3,\infty)$ *and* $k \in (0,\infty)$ *systems* (3.15) *have one singular point at* $q_1 = (0,0)$ *with eigenvalues* $\dfrac{1-a}{2}$ *and* $\dfrac{3-a}{2}$*. So* q_1 *is an hyperbolic unstable node if* $a \in (-\infty,1)$*, an hyperbolic stable node if* $a \in (3,\infty)$*, an hyperbolic saddle if* $a \in (1,3)$*, and a semi-hyperbolic singularity if* $a = 1$ *with eigenvalues* 0 *and* 1*. In order to obtain the local phase portrait at this point we use Theorem 2.19 of [7] and we obtain that* q_1 *is a saddle-node.*

If $a = 3$ *and* $k \in (0,\infty)$ *systems* (3.15) *become*

$$\dot{x} = -v\left(ku^2 + k - uv\right),$$
$$\dot{y} = v\left(-kuv + v^2 - 1\right).$$

These systems have infinity as a line of singularities. We take the change of variables $ds = vdt$*, we get the following systems*

$$\dot{x} = -ku^2 - k + uv, \quad \dot{y} = -kuv + v^2 - 1.$$

These systems have no singularity. Then the statement (ii) *holds.*

The differential systems (2.1) *in the local chart* U_2 *take the form:*

$$\dot{u} = \frac{1}{2}\left(a(u^2 - v^2 + 1) + u^2(2kv - 3) + 2kv + v^2 - 3\right),$$
$$\dot{v} = uv(kv - 1).$$

It is clear that the origin is not a singularity for this system for all $a \in \mathbb{R}$ *and* $k \in (0,\infty)$*.*

4 Local and global phase portraits

If $k \in (0,1)$*,* $a \neq 1$ *and from Proposition 3.3 we obtain the local phase portrait of the finite and infinite singular points. Due to the fact that the two singular points* q_1 *and* q_2 *belongs to the reducible invariant curve of the systems, we obtain some orbits on this invariant curve connecting those singular points, these connections vary if either* $a \in (-\infty,1) \cup (1,c_1) \cup (c_1,\infty)$*, or* $a \in (-\infty,1)$*, or* $a \in (c_1,c_2)$*, or* $a = c_1$*, or* $a = c_2$*, (see the local phase portraits* 1*, or* 2*, or* 3*, or* 4*, or* 5 *or* 7 *of Figure 3, respectively). Since* $\dot{x}_{|x=0} = \dfrac{1}{2}(a-3)y^2 + ky + \dfrac{1-a}{2}$*, and*

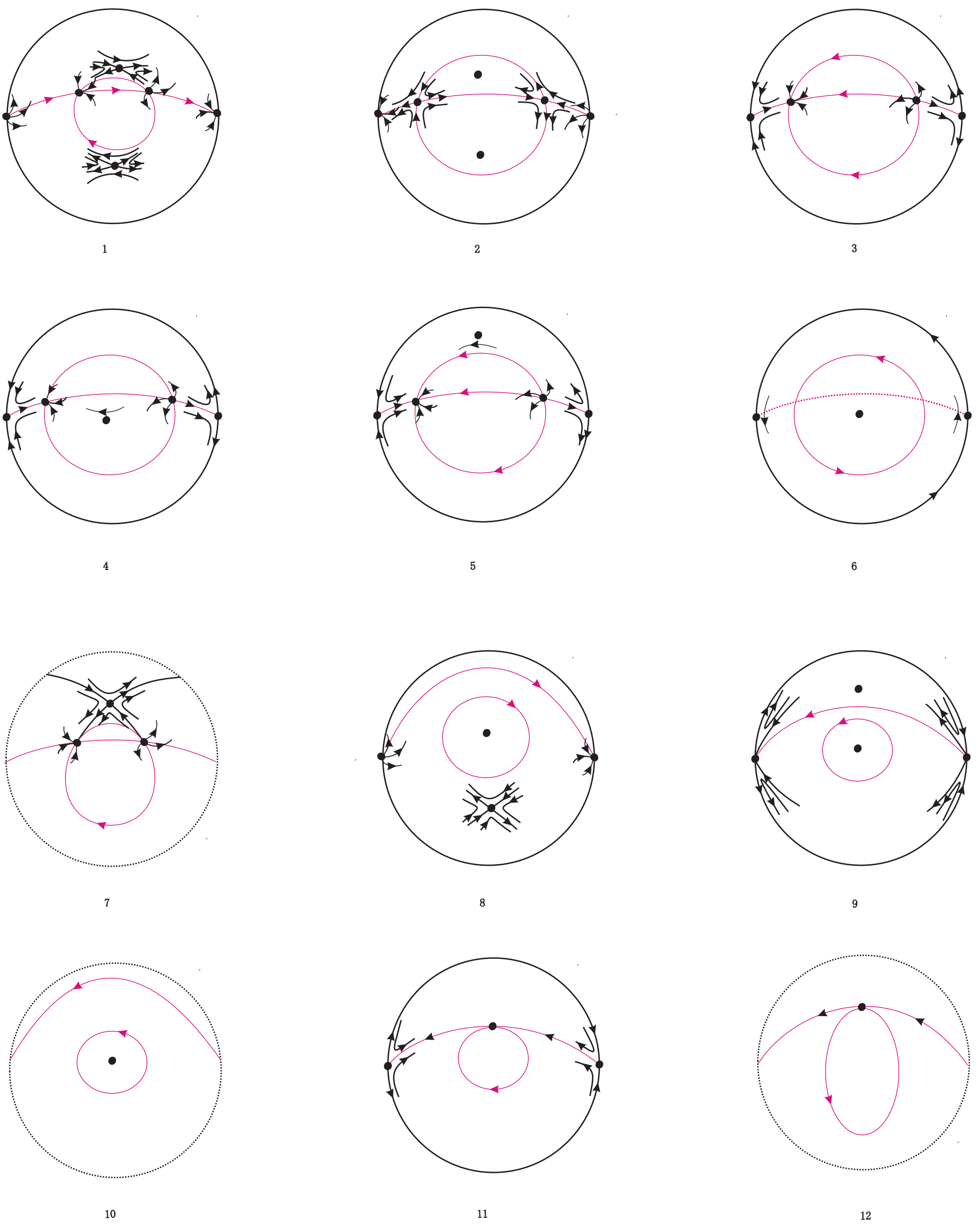

Figure 3. Local phase portraits at the singular points. The invariant algebraic curves of degree 3 are drawn in red color.

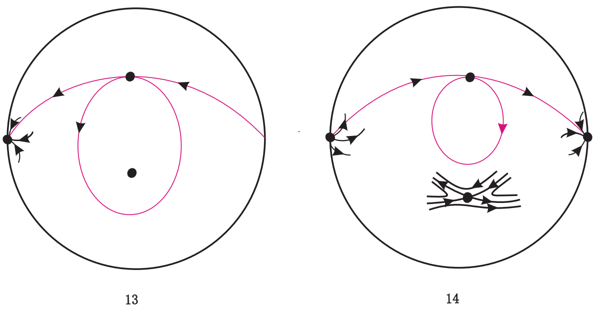

Figure 4. Continuation of Figure 3.

$\dot{y}_{|y=0} = -kx < 0$ *the separatrices for which we do not know their* α*- or* ω*-limit can be easily determined from the mentioned figures, obtaining the global phase portrait* 1*, or* 2*, or* 3*, or* 4*, or* 5 *or* 7 *of Figure 1.*

If $k \in (0, \infty)$ *and* $a = 1$ *the systems have* $y = k$ *as a line of singularities, and by doing the change of variables* $(y - k)dt = ds$ *we know that they have a center at the origin. Since* $\dot{x}_{|x=0} = y(k - y)$*, and* $\dot{y}_{|y=0} = 0$*, (see also the local phase portrait* 6 *of Figure 3) we get the global phase portrait* 6 *of Figure 1.*

If $k \in (1, \infty)$ *and* $a \neq 1$ *we get the local phase portraits* 8*,* 9 *and* 10 *of Figure 3 for the finite and infinite singular points of the systems. Since* $\dot{x}_{|x=0} = \frac{1}{2}(a-3)y^2 + ky + \frac{1-a}{2}$*, and* $\dot{y}_{|y=0} = -kx < 0$ *we get that the global phase portrait in this case is* 8*,* 9 *and* 10 *of Figure 1, respectively.*

If $k = 1$ *from Proposition 3.3 we obtain the local phase portrait of the finite and infinite singular points. Due to the fact that the singular point* p_3 *belongs to the reducible invariant curve of the system, we obtain some orbits on this invariant curve connecting this singular point, these connections vary in the interval* $a \in (-\infty, \infty)$*, (see local phase portraits* 11*, or* 12*, or* 13 *or* 14 *of Figure 3 and 4). Taking into account the following directions of the vector field of the systems in the axes* $\dot{x}_{|x=0} = \frac{1}{2}(a-3)y^2 + y + \frac{1-a}{2}$*, and* $\dot{y}_{|y=0} = -x < 0$ *we get to the global phase portrait* 11*, or* 12*, or* 13 *or* 14 *of Figure 1 and 2.*

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Received: June 24, 2021.

Accepted: September 15, 2021.