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Numerical solutions for linear fractional differential equation with dependence on the Caputo-Hadamard derivative using finite difference method

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Abstract The main objective of this paper is to find accurate solutions for linear fractional differential equations involving the fractional Caputo-Hadamard derivative of order $\alpha > 0$. Therefore, to achieve this objective, a new method called the Finite Fractional Difference Method (FFDM) is employed to find the numerical solution. As such, the convergence and stability of the numerical scheme is discussed and illustrated by solving two linear fractional differential equation problems of order $0 < \alpha \leqslant 1$ to show the validity of our method.

1 Introduction

Fractional calculus had played a very important role in in diverse fields of science and engineering. Due to its richness in theory and its various applications, many researchers have been interested in this field, for details, see (Samko et al. 1993 [19], Podlubny 1999 [16], Kilbas et al. 2006 [7], Diethelm 2010 [4]). Fractional differential equations have been used in the study of models of many phenomena in various fields of sciences, physics [6], fractional signal processing techniques [20] and many others areas (see e.g. [18]), However, among the investigations for fractional differential equations, the search for exact and numerical solutions of fractional differential equations. Many methods have been proposed to obtain numerical and exact solutions of fractional differential equations. For example, these methods include the laplace transform method, the adomian decomposition method, the fourier transform method, the homotopy perturbation method, the differential transformation method and so on. In these investigations, it should be noted that the determination of exact solutions is not an easy task, even inaccessible for some nonlinear fractional differential equations. For more details see ([2]-[3]-[5]-[13]-[15]-[21]).

This paper concerns the numerical solution for fractional differential equation of Caputo-Hadamard type given by:

$$\begin{cases} {}^{CH}\mathcal{D}^{\alpha}_{a^+}u\,(t) + c(t)u(t) = f(t),\ 0 < a \le t \le b < \infty, \\ u(a) = u_0, \end{cases} \tag{1.1}$$

where ${}^{CH}\mathcal{D}^{\alpha}$ denotes the Caputo-Hadamard fractional derivative operator of order α between zero and one ([1]-[8]), defined by

$${}^{CH}\mathcal{D}^{\alpha}_{a^+}u\,(t) = \frac{1}{\Gamma(1-\alpha)}\int_a^t \left(\log\frac{t}{s}\right)^{-\alpha}\left(s\frac{d}{ds}\right)u(s)\frac{ds}{s}.$$

The rest of this paper is organized as follows. In Section 2, we present some definitions and properties of the Caputo-Hadamard fractional integrals and fractional derivatives of various types. In Section 3 finite difference methods (FDM) for the problem is presented. Using the discrete implicit Euler formula we obtain an approximate sequence for (1.1). Convergence and stability for the (FFDM) are discussed in 4. In Section 5, two numerical examples are presented to verify the accuracy and efficiency of the proposed scheme. Section 6, conclusion close the paper.

2 Preliminaries

In this section, we recall some concepts on fractional calculus and present additional properties that will be used later.

Definition 2.1 (Hadamard fractional integral). (see [7])
The left-sided Hadamard fractional integral of order $\alpha > 0$ of a function $y : (a, b) \to \mathbb{R}$ is given by

$$\mathcal{I}_{a^+}^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{s}\right)^{\alpha-1} y(s) \frac{ds}{s}, \tag{2.1}$$

provided the right integral converges.

Similarly we can define right-sided integrals [7].

Definition 2.2 (Hadamard fractional derivative). (see [7]).
The left-sided Hadamard fractional derivative of order $\alpha \geq 0$ of a continuous function $y : (a, b) \to \mathbb{R}$ is given by

$$\mathcal{D}_{a^+}^{\alpha} f(t) = \delta^n \mathcal{I}_{a^+}^{n-\alpha} = \frac{1}{\Gamma(n-\alpha)} \left(t \frac{d}{dt}\right)^n \int_a^t \left(\log \frac{t}{s}\right)^{n-\alpha-1} y(s) \frac{ds}{s}, \tag{2.2}$$

provided the right integral converges,
where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of the real number α and $\delta = t\dfrac{d}{dt}$.

A recent generalization introduced by Jarad and al in [8]. The authors define the generalization of the Hadamard fractional derivatives and present properties of such derivatives. This new generalization is now know as the Caputo-Hadamard fractional derivatives and is given by the following definition:

Definition 2.3 (Caputo-Hadamard fractional derivative). (see [8]).
The left and right sided Hadamard fractional derivatives of order α are respectively defined by

$$^{CH}\mathcal{D}_{a^+}^{\alpha} y(t) = \mathcal{I}_{a^+}^{n-\alpha} \delta^n y(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \left(\log \frac{t}{s}\right)^{n-\alpha-1} \delta^n y(s) \frac{ds}{s}. \tag{2.3}$$

$$^{CH}\mathcal{D}_{b^-}^{\alpha} y(t) = \mathcal{I}_{b^-}^{n-\alpha} \delta^n y(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_t^b \left(\log \frac{t}{s}\right)^{n-\alpha-1} \delta^n y(s) \frac{ds}{s}. \tag{2.4}$$

Property 2.4. (see [8]). Let $\Re(\alpha) \geq 0$, and $n = [\Re(\alpha)] + 1$ and $\Re(\beta) > 0$. Then

(i) $^{CH}\mathcal{D}_{a^+}^{\alpha} \left(\log\left(\frac{t}{a}\right)\right)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} \left(\log\left(\frac{t}{a}\right)\right)^{\beta-\alpha-1} \quad \Re(\beta) > n,$

(ii) $^{CH}\mathcal{D}_{b^-}^{\alpha} \left(\log\left(\frac{b}{t}\right)\right)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} \left(\log\left(\frac{b}{t}\right)\right)^{\beta-\alpha-1} \quad \Re(\beta) > n,$

(iii) $^{CH}\mathcal{D}_{a^+}^{\alpha} \left(\log\left(\frac{t}{a}\right)\right)^{k} = 0, \ ^{CH}\mathcal{D}_{b^-}^{\alpha} \left(\log\left(\frac{b}{t}\right)\right)^{k} = 0, \ k = 0, 1, ..., n-1,$

(iv) $^{CH}\mathcal{D}_{a^+}^{\alpha} 1 = 0, \ ^{CH}\mathcal{D}_{b^-}^{\alpha} 1 = 0.$

3 The Fractional Finite Difference Method (FFDM)

For the finite difference approximation, we equally sub-divide the intervals $[a, T]$ with $t_i = a + ih$, $i = 0, 1, \ldots, N$, where $h = \dfrac{T-a}{N}$ is the step size.

Let $u : [a, T] \to \mathbb{R}$ be a given function, u_n the numerical approximation of u at points t_n and $f_n = f(t_n)$, our result is presented as follows.

Theorem 3.1. *Let* $u : [a,T] \to \mathbb{R}$ *such that* $u \in C^2([a,T], \mathbb{R})$, α *between zero and one, then for* $N \in \mathbb{N}$, *we have*

$$^{CH}\mathcal{D}^{\alpha}_{a^+} u(t_n) = {}^{CH}\mathcal{D}^{\alpha}_{a^+} u_n + O\left(h^{1-\alpha}\right),$$

where $^{CH}\mathcal{D}^{\alpha}_{a^+} u_n$ *is defined as follows:*

$$^{CH}\mathcal{D}^{\alpha}_{a^+} u_n = \frac{1}{h\Gamma(2-\alpha)} \sum_{i=1}^{n} b_i \left(u(t_i) - u(t_{i-1})\right), \tag{3.1}$$

and

$$b_i = t_i \left(\left(\log \frac{t_n}{t_{i-1}}\right)^{1-\alpha} - \left(\log \frac{t_n}{t_i}\right)^{1-\alpha} \right). \tag{3.2}$$

Proof. For any $N \in \mathbb{N}$ and for each $n \in \{0, 1, \dots, N\}$, we have

$$\begin{aligned}
^{CH}\mathcal{D}^{\alpha}_{a^+} u(t_n) =& \frac{1}{\Gamma(1-\alpha)} \int_a^{t_n} \left(\log \frac{t_n}{s}\right)^{-\alpha} \left(s\frac{d}{ds}\right) u(s) \frac{ds}{s} \\
\approx& \frac{1}{\Gamma(1-\alpha)} \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \left(\log \frac{t_n}{s}\right)^{-\alpha} t_i \left(\frac{u(t_i) - u(t_{i-1})}{t_i - t_{i-1}}\right) \frac{ds}{s} \\
=& \frac{1}{\Gamma(1-\alpha)} \sum_{i=1}^{n} t_i \left(\frac{u(t_i) - u(t_{i-1})}{h}\right) \int_{t_{i-1}}^{t_i} \left(\log \frac{t_n}{s}\right)^{-\alpha} \frac{ds}{s} \\
=& \frac{1}{h\Gamma(1-\alpha)} \sum_{i=1}^{n} t_i \left(u(t_i) - u(t_{i-1})\right) \left[- \frac{\left(\log \frac{t_n}{s}\right)^{1-\alpha}}{(1-\alpha)} \right]_{t_{i-1}}^{t_i} \\
=& \frac{1}{h\Gamma(2-\alpha)} \sum_{i=1}^{n} t_i \left[\left(\log \frac{t_n}{t_{i-1}}\right)^{1-\alpha} - \left(\log \frac{t_n}{t_i}\right)^{1-\alpha} \right] \left(u(t_i) - u(t_{i-1})\right) \\
=& \frac{1}{h\Gamma(2-\alpha)} \sum_{i=1}^{n} b_i \left(u(t_i) - u(t_{i-1})\right) \\
=& {}^{CH}\mathcal{D}^{\alpha}_{a^+} u_n .
\end{aligned}$$

Set $E_n = \left| ^{CH}\mathcal{D}^{\alpha}_{a^+} u(t_n) - {}^{CH}\mathcal{D}^{\alpha}_{a^+} u_n \right|$ and $M_i = max \left| u^{(i)}(t) \right|$, $i = 1, 2$, hence, we can obtain

$$E_n \leq \frac{1}{\Gamma(1-\alpha)} \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \left(\log \frac{t_n}{s}\right)^{-\alpha} \left| s\frac{du}{ds} - t_i \left(\frac{u(t_i) - u(t_{i-1})}{t_i - t_{i-1}}\right) \right| \frac{ds}{s}.$$

It follows from Taylor's theorem, one has for each $i \in \{1, \dots, N\}$, with $s \in [t_{i-1}, t_i]$ and $\eta_1 \in [t_{i-1}, t_i]$, $\eta_2 \in [t_{i-1}, s]$

$$\begin{aligned}
\left| s\frac{du}{ds} - t_i \left(\frac{u(t_i) - u(t_{i-1})}{t_i - t_{i-1}}\right) \right| =& \left| s\frac{du}{ds} - t_i \left(\frac{du(t_{i-1})}{ds} - \frac{d^{(2)}u(\eta_1)}{ds^2} \frac{h}{2!} \right) \right| \\
\leq& \left| \left(s\frac{du}{ds} - t_i \frac{du(t_{i-1})}{ds} \right) \right| + M_2 \frac{t_i h}{2} \\
=& \left| s \left(\frac{du(t_{i-1})}{ds} - t_i \frac{du(t_{i-1})}{ds} + \frac{d^{(2)}u(\eta_1)}{ds^2} (s - t_{i-1}) \right) \right| + M_2 \frac{t_i h}{2} \\
\leq& M_1 (t_i - t_{i-1}) + M_2 t_i \frac{3}{2} h \\
\leq& \left(M_1 + \frac{3T}{2} M_2 \right) h.
\end{aligned}$$

Furthermore, for any $0 < \alpha \leq 1$ and $n \in \{1, \ldots, N\}$ with $i \leq n$, $s \in [t_{i-1}, t_i]$

$$0 \leq \left(\log \frac{t_n}{s}\right)^{-\alpha} \leq \left(\log \frac{t_i}{s}\right)^{-\alpha},$$

Therefore, we conclude

$$\begin{aligned}
E_n \leq & \frac{1}{\Gamma(1-\alpha)}\left(M_1 + \frac{3T}{2}M_2\right) h \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \left(\log \frac{t_n}{s}\right)^{-\alpha} \frac{ds}{s} \\
\leq & \frac{1}{\Gamma(1-\alpha)}\left(M_1 + \frac{3T}{2}M_2\right) h \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \left(\log \frac{t_i}{s}\right)^{-\alpha} \frac{ds}{s} \\
\leq & \frac{1}{\Gamma(2-\alpha)}\left(M_1 + \frac{3T}{2}M_2\right) h \sum_{i=1}^{n} \left(\log \frac{t_i}{t_{i-1}}\right)^{1-\alpha} \\
\leq & \frac{1}{\Gamma(2-\alpha)}\left(M_1 + \frac{3T}{2}M_2\right) h \sum_{i=1}^{n} \left(\frac{t_i}{t_{i-1}}\right)^{1-\alpha} \\
\leq & \frac{1}{\Gamma(2-\alpha)}\left(M_1 + \frac{3T}{2}M_2\right) h \sum_{i=1}^{N} h^{1-\alpha} T^{1-\alpha} \\
\leq & \frac{1}{\Gamma(2-\alpha)}\left(M_1 + \frac{3T}{2}M_2\right) T^{1-\alpha} h^{1-\alpha} \sum_{i=1}^{N} h \\
\leq & \frac{1}{\Gamma(2-\alpha)}\left(M_1 + \frac{3T}{2}M_2\right) (T-a)\, T^{1-\alpha} h^{1-\alpha},
\end{aligned}$$

which means

$${}^{CH}\mathcal{D}_{a^+}^{\alpha} u\,(t_n) = {}^{CH}\mathcal{D}_{a^+}^{\alpha} u_n + c_\alpha h^{1-\alpha}.$$

□

Now, by using the fractional approximation (3.1), we obtain the following numerical approximation of the problem (1.1)

$$\frac{1}{h\Gamma(2-\alpha)} \sum_{i=1}^{n} b_i \left(u_i - u_{i-1}\right) + c_n u_n = f(t_n), \tag{3.3}$$

the resulting equation can be written as

$$\left(\frac{b_n + h\Gamma(2-\alpha)c_n}{h\Gamma(2-\alpha)}\right) u_n = f(t_n) + \frac{1}{h\Gamma(2-\alpha)} b_n u_{n-1} - \frac{1}{h\Gamma(2-\alpha)} \sum_{i=1}^{n-1} b_i \left(u_i - u_{i-1}\right),$$

witch gives

$$u_n = \left(\frac{b_n}{\omega_n}\right) u_{n-1} - \left(\frac{1}{\omega_n}\right) \sum_{i=1}^{n-1} b_i \left(u_i - u_{i-1}\right) + \left(\frac{\lambda}{\omega_n}\right) f(t_n), \tag{3.4}$$

the above equation can be rewritten as the following form

$$u_n = \frac{b_1}{\omega_n} u(t_0) + \frac{1}{\omega_n} \sum_{i=1}^{n-2} G_i u_i + \left(\frac{b_n - b_{n-1}}{\omega_n}\right) u_{n-1} + \frac{\lambda}{\omega_n} f(t_n), \tag{3.5}$$

with

$$\begin{cases} u(t_0) = u(a), \\ \omega_n = \left(b_n + \lambda c_n\right), \\ G_i = b_{i+1} - b_i, \\ \lambda = h\Gamma(2-\alpha). \end{cases}$$

4 Stability and Convergence of Finite Difference Method (FFDM)

In this section, we discuss the stability and the convergence of the finite difference scheme (3.5) for the fractional differential equation (1.1). For that, we need the following lemma

Lemma 4.1. *For $n = 1, 2, \dots, N$, the coefficients b_n in (3.2) satisfy*

1. $b_n > 0$, *for* $n = 1, 2, \dots, N$.

2. $b_n > b_{n-1}$, *for* $i = 2, \dots, N$.

Firstly, we consider the stability of the difference approximation (3.5). We suppose that u_n and $u(t_n)$ are the approximate and the exact solution of (3.5) respectively for $n = 1, 2, \dots, N$. Set $\varepsilon^n = u_n - u(t_n)$ then, from (3.5) we have

$$\varepsilon^n = \frac{b_1}{\omega_n}\varepsilon^0 + \frac{1}{\omega_n}\sum_{i=1}^{n-2} G_i\varepsilon^i + \left(\frac{b_n - b_{n-1}}{\omega_n}\right)\varepsilon^{n-1},$$

which can be written as

$$E^n = \frac{b_1}{\omega_n}E^0 + \frac{1}{\omega_n}\sum_{i=1}^{n-2} G_iE^i + \left(\frac{b_n - b_{n-1}}{\omega_n}\right)E^{n-1}.$$

Hence, the following result can be proved.

Lemma 4.2. *The stability condition is equivalent to*

$$\|E^n\|_\infty \le \left\|E^0\right\|_\infty, \ \textit{for } n = 1, 2, 3, \dots, N.$$

Proof. We will use mathematical induction to get the above result. For $n = 1$ and because $\frac{b_1}{\omega_1} \le 1$, we have

$$\begin{aligned}\left\|E^1\right\|_\infty = \left|\varepsilon^1\right| &\le \frac{b_1}{\omega_1}\left\|E^0\right\|_\infty \\ &\le \left\|E^0\right\|_\infty.\end{aligned}$$

Suppose that $\left\|E^i\right\|_\infty \le \left\|E^0\right\|_\infty$ for $i = 1, 2, 3, \dots, n-1$, using lemma (4.1) we get

$$\begin{aligned}\|E^n\|_\infty = |\varepsilon^n| &\le \frac{b_1}{\omega_n}\left\|E^0\right\|_\infty + \frac{1}{\omega_n}\sum_{i=1}^{n-2}|G_i|\left\|E^i\right\|_\infty + \left|\frac{b_n - b_{n-1}}{\omega_n}\right|\left\|E^{n-1}\right\|_\infty \\ &\le \frac{b_1}{\omega_n}\left\|E^0\right\|_\infty + \frac{1}{\omega_n}\sum_{i=1}^{n-2}|b_{i+1} - b_i|\left\|E^i\right\|_\infty + \left|\frac{b_n - b_{n-1}}{\omega_n}\right|\left\|E^0\right\|_\infty \\ &\le \frac{b_1}{\omega_n}\left\|E^0\right\|_\infty + \frac{1}{\omega_n}(b_{n-1} - b_1)\left\|E^0\right\|_\infty + \frac{b_n - b_{n-1}}{\omega_n}\left\|E^0\right\|_\infty \\ &\le \frac{b_n}{\omega_n}\left\|E^0\right\|_\infty \\ &\le \left\|E^0\right\|_\infty.\end{aligned}$$

Hence, the proof is completed. □

Secondly, we consider the convergence of the difference approximation (3.5). Define $e^n = u(t_n) - u^n$ using $e^0 = 0$, subtiting $u^n = u(t_n) - e^n$ into (3.5) leads to:

$$\begin{aligned}(u(t_n) - e^n) = \left(\frac{b_n - b_{n-1}}{\omega_n}\right)\left(u(t_{n-1}) - e^{n-1}\right) + \frac{1}{\omega_n}\sum_{i=1}^{n-2} G_i\left(u(t_i) - e^i\right) \\ + \frac{b_1}{\omega_n}\left(u(t_0) - e^0\right) + \frac{\lambda}{\omega_n}f_n,\end{aligned}$$

then, we get

$$\begin{aligned} e^n =& u(t_n) - \frac{b_1}{\omega_n} u(t_0) - \frac{1}{\omega_n} \sum_{i=1}^{n-2} G_i u(t_i) - \left(\frac{b_n - b_{n-1}}{\omega_n}\right) u(t_{n-1}) - \frac{\lambda}{\omega_n} f\left(t_n\right) \\ &+ \left(\frac{b_n - b_{n-1}}{\omega_n}\right) e^{n-1} + \frac{1}{\omega_n} \sum_{i=1}^{n-2} G_i e^i + \frac{b_1}{\omega_n} e^0 \\ =& \left(\frac{b_n - b_{n-1}}{\omega_n}\right) e^{n-1} + \frac{1}{\omega_n} \sum_{i=1}^{n-2} G_i e^i + \frac{b_1}{\omega_n} e^0 + R^n, \end{aligned}$$

where

$$\begin{aligned} R^n &= \left(\sum_{i=1}^{n} b_i \left(u(t_i) - u\left(t_{i-1}\right)\right) + \lambda c(t_n) u\left(t_n\right) - \lambda f\left(t_n\right) \right) \\ &= h\Gamma(2-\alpha) \left({}^{CH}\mathcal{D}_{a^+}^{\alpha} u\left(t_n\right) + c(t_n) u\left(t_n\right) - f\left(t_n\right) + c_\alpha h^{1-\alpha} \right) \\ &= c_\alpha \Gamma(2-\alpha) h^{2-\alpha}. \end{aligned}$$

Hence, there exist c'_α such that

$$|R^n| \leq c'_\alpha h^{2-\alpha}.$$

Consequently, using mathematical induction, we prove

$$\|e^n\| \leq C_\alpha h^{2-\alpha}.$$

For $n = 1$, we get

$$\begin{aligned} \left\|e^1\right\| &\leq \left|R^1\right| \\ &\leq c'_\alpha h^{2-\alpha}. \end{aligned}$$

Suppose that $\left\|e^i\right\| \leq c'_\alpha h^{2-\alpha}$. for $i = 1, 2, \ldots, n-1$, using lemma (4.1), we have

$$\begin{aligned} \|e^n\| &\leq \left| \frac{b_1}{\omega_n} e^0 + \frac{1}{\omega_n} \sum_{i=1}^{n-2} G_i e^i + \left(\frac{b_n - b_{n-1}}{\omega_n}\right) e^{n-1} + R^n \right| \\ &\leq \frac{1}{\omega_n} \sum_{i=1}^{n-2} \left(b_{i+1} - b_i\right) e^i + \left(\frac{b_n - b_{n-1}}{\omega_n}\right) \left|e^{n-1}\right| + \left|R^n\right| \\ &\leq \left(\frac{b_{n-1} - b_1}{\omega_n}\right) c'_\alpha h^{2-\alpha} + \left(\frac{b_n - b_{n-1}}{\omega_n}\right) c'_\alpha h^{2-\alpha} + c'_\alpha h^{2-\alpha} \\ &\leq 2c'_\alpha h^{2-\alpha} \\ &\leq C_\alpha h^{2-\alpha}. \end{aligned}$$

Hence, the following theorem is obtained and guarantees the stability and convergence of the discretized scheme.

Theorem 4.3. *The obtained approximation sequence u_n, for the discretized scheme (3.5) is stable and convergent, if $C_\alpha h^{2-\alpha}$ tends to zero.*

5 Illustrative examples

In this section, we present two examples to illustrate the usefulness of our main results.

Example 5.1. Let $t \in [1, 2]$ and $\alpha = 0.8$ and

$$f(t) = \frac{1}{\Gamma\left(2-\alpha\right)} \left(\log t\right)^{1-\alpha} + t \log\left(\frac{t}{3}\right)$$

Consider the following generalized Caputo–Hadamard fractional differential equation:

$$\begin{cases} {}^{CH}\mathcal{D}_{a^+}^{\alpha} u\left(t\right) + tu(t) = f(t), \ 1 \le t \le 2, \\ u(1) = \log\left(\dfrac{1}{3}\right). \end{cases} \tag{5.1}$$

The exact solution of (5.1) is given by:

$$u(t) = \log\left(\frac{t}{3}\right).$$

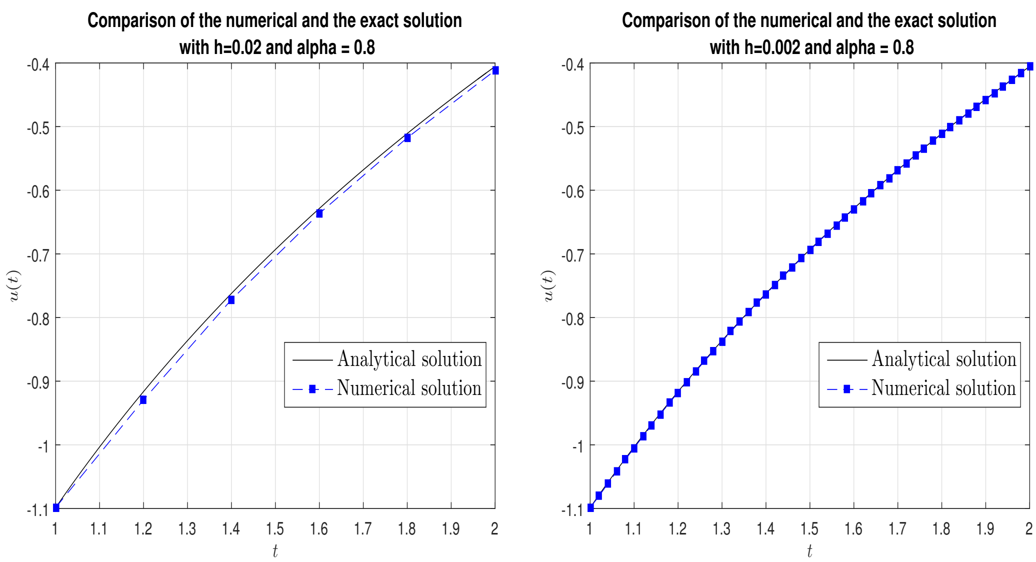

Figure 1. Graphical comparison of the numerical and the exact solution.

t	Exact solution	Approx solution	Error for $h = 0.02$
1.0	-1.09861	-1.09861	0.00000e+00
1.1	-1.00330	-1.01907	1.57665e-02
1.2	-0.91629	-0.92935	1.30629e-02
1.3	-0.83625	-0.84738	1.11362e-02
1.4	-0.76214	-0.77186	9.71583e-03
1.5	-0.69315	-0.70180	8.65184e-03
1.6	-0.62861	-0.63646	7.84803e-03
1.7	-0.56798	-0.57522	7.23832e-03
1.8	-0.51083	-0.51760	6.77543e-03
1.9	-0.45676	-0.46318	6.42460e-03
2.0	-0.40547	-0.41162	6.15984e-03

t	Exact solution	Approx solution	Error for $h = 0.002$
1.0	-1.09861	-1.09861	0.00000e+00
1.1	-1.00330	-1.00486	1.56088e-03
1.2	-0.91629	-0.91759	1.29508e-03
1.3	-0.83625	-0.83735	1.10569e-03
1.4	-0.76214	-0.76311	9.66025e-04
1.5	-0.69315	-0.69401	8.61377e-04
1.6	-0.62861	-0.62939	7.82305e-04
1.7	-0.56798	-0.56871	7.22324e-04
1.8	-0.51083	-0.51150	6.76787e-04
1.9	-0.45676	-0.45740	6.42279e-04
2.0	-0.40547	-0.40608	6.16243e-04

Table 1. Comparison of the numerical and the exact solutions with $h = 0.02$, $h = 0.002$, and $\alpha = 0.8$.

Example 5.2. Let $t \in [1, 3]$ and $\alpha = 0.5$ and

$$f(t) = (t+1)\log\left(\frac{\sqrt{2}}{t}\right) - \frac{1}{\Gamma\left(2-\alpha\right)}\left(\log t\right)^{1-\alpha}$$

Consider the following generalized Caputo–Hadamard fractional differential equation:

$$\begin{cases} {}^{CH}\mathcal{D}_{a^+}^{\alpha} u\left(t\right) + (t+1)u(t) = f(t), \ 1 \le t \le 3 \\ u(1) = \log\left(\sqrt{2}\right). \end{cases}, \tag{5.2}$$

The exact solution of this problem is given by:

$$u(t) = \log\left(\frac{\sqrt{2}}{t}\right).$$

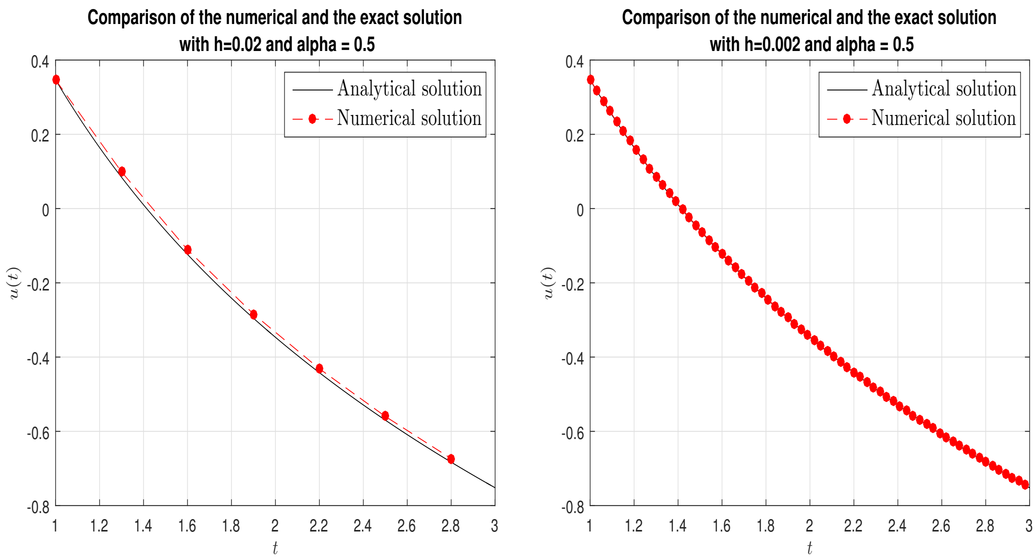

Figure 2. Graphical comparison of the numerical and the exact solution.

t	Exact solution	Approx solution	Error for $h = 0.02$	t	Exact solution	Approx solution	Error for $h = 0.002$
1.0	0.34657	0.34657	0.00000e+00	1.0	0.34657	0.34657	0.00000e+00
1.2	0.16425	0.17993	1.56789e-02	1.2	0.16425	0.16581	1.55983e-03
1.4	0.01010	0.02393	1.38249e-02	1.4	0.01010	0.01148	1.37844e-03
1.6	-0.12343	-0.11080	1.26291e-02	1.6	-0.12343	-0.12217	1.26056e-03
1.8	-0.24121	-0.22943	1.17796e-02	1.8	-0.24121	-0.24004	1.17651e-03
2.0	-0.34657	-0.33544	1.11379e-02	2.0	-0.34657	-0.34546	1.11285e-03
2.2	-0.44188	-0.43125	1.06299e-02	2.2	-0.44188	-0.44082	1.06237e-03
2.4	-0.52890	-0.51868	1.02127e-02	2.4	-0.52890	-0.52787	1.02085e-03
2.6	-0.60894	-0.59908	9.85944e-03	2.6	-0.60894	-0.60795	9.85646e-04
2.8	-0.68305	-0.67349	9.55292e-03	2.8	-0.68305	-0.68209	9.55076e-04
3.0	-0.75204	-0.74276	9.28159e-03	3.0	-0.75204	-0.75111	9.27997e-04

Table 2. Comparison of the numerical and the exact solutions with $h = 0.02$, $h = 0.002$, and $\alpha = 0.5$.

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6 Conclusion

In this paper we have developed a fractional finite difference Method (FFDM) for a generalized fractional differential equation of Caputo-Hadamard type. Also, we have proved that the approximate solution u_n is stable and convergent. The efficiency of (FFDM) has been discussed and illustrated by solving two typical examples (Example 5.1 and Example 5.2). it is found that the approximate solutions produced by this method are in complete agreement with the corresponding exact solutions (Figure 1, Figure 2). The results obtained show a good global approximation and an improved convergence with an error $C_\alpha(h^{2-\alpha})$ reaching to zero. (Table 1, Table 2).

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