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ON AMPLY g-SUPPLEMENTED LATTICES

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Abstract In this work, all lattices are complete modular lattices with the greatest element 1 and the smallest element 0. Let L be a lattice and $a \in L$. If for every $b \in L$ such that $a \vee b = 1$, a has a g-supplement x in L such that $x \leq b$, then we say a has ample g-supplements in L. If every element of L has ample g-supplements in L, then L is called an amply g-supplemented lattice (See [7]). In this work, some properties of these lattices are investigated.

1 Introduction

Throughout this work all lattices are complete modular lattices with the smallest element 0 and the greatest element 1. Let L be a lattice, $a, b \in L$ and $a \leq b$. A sublattice $\{x \in L | a \leq x \leq b\}$ is called a *quotient* sublattice of L, denoted by b/a. Let L be a lattice and $a \in L$. If $a \wedge t \neq 0$ for every element $t \neq 0$ of L, then a is called an *essential* element of L and denoted by $a \trianglelefteq L$. $a \trianglelefteq L$ if and only if $t = 0$ for every $t \in L$ with $a \wedge t = 0$. Let L be a lattice and $a \in L$. If $t = 1$ for every element t of L with $a \vee t = 1$, then a is called a *small* (or *superfluous*) element of L and denoted by $a \ll L$. The meet of all maximal $(\neq 1)$ elements of L is called the *radical* of L and denoted by $r(L)$. If L have no maximal $(\neq 1)$ elements, then we call $r(L) = 1$. Let L be a lattice and $a, b \in L$. If $a \vee b = 1$ and b is minimal with respect to this property, or equivalently, $a \vee b = 1$ and $a \wedge b \ll b/0$, then b is called a *supplement* of a in L. If every element of L has a supplement in L, then L is called a *supplemented* lattice. Let L be a lattice and $a \in L$. If for every $b \in L$ with $a \vee b = 1$, a has a supplement x in L with $x \leq b$, then we say a has *ample supplements* in L. If every element of L has ample supplements in L, then L is called an *amply supplemented* lattice. Let L be a lattice. If $b \ll L$ for every element b of L with $b \neq 1$, then L is called a *hollow* lattice. If L has the greatest element $(\neq 1)$, then L is called a *local* lattice. Let L be a lattice and $a \in L$. If $t = 1$ for every essential element t of L with $a \vee t = 1$, then a is called a *generalized small* (or briefly, *g-small*) element of L and denoted by $a \ll_g L$. Let L be a lattice and $a, b \in L$. b is called a *g-supplement* of a in L if $a \vee b = 1$ and $t = b$ for every $t \trianglelefteq b/0$ with $a \vee t = 1$. b is a g-supplement of a in L if and if $a \vee b = 1$ and $a \wedge b \ll_g b/0$. L is said to be *g-supplemented* if every element of L has a g-supplement in L.

More details about (amply) supplemented lattices are in [1] and [3]. Supplemented and amply supplemented modules are studied in [2] and [9]. G-supplemented modules are studied in [4] and [5]. The definition of g-supplemented lattices and some properties of these lattices are in [8].

Lemma 1.1. *Let L be a lattice and $a, b, c, d \in L$. Then the followings hold.*
(i) *If $a \leq b$ and $b \ll_g L$, then $a \ll_g L$.*
(ii) *If $a \ll_g b/0$, then $a \ll_g t/0$ for every $t \in L$ with $b \leq t$.*
(iii) *If $a \ll_g L$, then $a \vee b \ll_g 1/b$.*
(iv) *If $a \ll_g b/0$ and $c \ll_g d/0$, then $a \vee c \ll_g (b \vee d)/0$.*

Proof. See [8, Lemma 1]. □

2 Amply g-Supplemented Lattices

Definition 2.1. Let L be a lattice and $a \in L$. If for every $b \in L$ such that $a \vee b = 1$, a has a g-supplement x in L such that $x \leq b$, then we say a has ample g-supplements in L. If every element of L has ample g-supplements in L, then L is called an amply g-supplemented lattice (See [7]).

Lemma 2.2. *Every amply supplemented lattice is amply g-supplemented.*

Proof. Let L be an amply supplemented lattice and $a \vee b = 1$ with $a, b \in L$. Since L is amply supplemented, a has a supplement x in L with $x \leq b$. Since x is a supplement of a in L, $1 = a \vee x$ and $a \wedge x \ll x/0$. Since $a \wedge x \ll x/0$, $a \wedge x \ll_g x/0$. Hence x is a g-supplement of a in L. Thus L is amply g-supplemented. □

Corollary 2.3. *Let L be an amply supplemented lattice. Then $1/a$ is amply g-supplemented for every $a \in L$.*

Proof. Since L is amply supplemented, $1/a$ is amply supplemented. Then by Lemma 2.2, $1/a$ is amply g-supplemented. □

Corollary 2.4. *Let L be a finite lattice. Then L is amply g-supplemented.*

Proof. Since L is finite, L is amply supplemented. Then by Lemma 2.2, L is amply g-supplemented. □

Proposition 2.5. *Let L be an amply g-supplemented lattice. If every nonzero element of L is essential in L, then L is amply supplemented.*

Proof. Clear from definitions. □

Proposition 2.6. *Let L be an amply g-supplemented lattice. If every nonzero element of L is essential in L, then $1/a$ is amply supplemented for every $a \in L$.*

Proof. Since L is amply g-supplemented and every nonzero element of L is essential in L, by Proposition 2.5, L is amply supplemented. Since L is amply supplemented, $1/a$ is amply supplemented for every $a \in L$. □

Proposition 2.7. *Let L be an amply g-supplemented lattice. If every nonzero element of L is essential in L, then L is supplemented.*

Proof. Since L is amply g-supplemented and every nonzero element of L is essential in L, by Proposition 2.5, L is amply supplemented. Since L is amply supplemented, L is supplemented too. □

Lemma 2.8. *Let M be an amply supplemented $R-$module and A be the family of all submodules of M. Then A is an amply supplemented lattice by the operation $\subset$.*

Proof. Clear from definitions. □

Corollary 2.9. *Let M be an amply supplemented $R-$module and A be the family of all submodules of M. Then A is an amply g-supplemented lattice by the operation $\subset$.*

Proof. Since M is an amply supplemented $R-$module, by Lemma 2.8, A is an amply supplemented lattice. Then by Lemma 2.2, A is amply g-supplemented. □

Corollary 2.10. *Let M be a $\pi-$projective and supplemented $R-$module and A be the family of all submodules of M. Then A is an amply supplemented lattice by the operation $\subset$.*

Proof. Since M is a $\pi-$projective and supplemented $R-$module, by [9, 41.15], M is an amply supplemented $R-$module. Then by Lemma 2.8, A is an amply supplemented lattice. □

Corollary 2.11. *Let M be a $\pi-$projective and supplemented $R-$module and A be the family of all submodules of M. Then A is an amply g-supplemented lattice by the operation $\subset$.*

Proof. By Corollary 2.10, A is an amply supplemented lattice. Then by Lemma 2.2, A is amply g-supplemented. □

Lemma 2.12. *Let M be an $R-$module and A be the family of all submodules of M. If every submodule of M lies above a direct summand in M, then A is an amply supplemented lattice by the operation $\subset$.*

Proof. Since every submodule of M lies above a direct summand in M, by [9, 41.15], M is an amply supplemented $R-$module. Then by Lemma 2.8, A is an amply supplemented lattice. □

Corollary 2.13. *Let M be an $R-$module and A be the family of all submodules of M. If every submodule of M lies above a direct summand in M, then A is an amply g-supplemented lattice by the operation $\subset$.*

Proof. By Lemma 2.12, A is an amply supplemented lattice. Then by Lemma 2.2, A is amply g-supplemented. □

Lemma 2.14. *Let M be a $\pi-$projective $R-$module and A be the family of all submodules of M. If every submodule of M is β^* equivalent to a supplement submodule in M, then A is an amply supplemented lattice by the operation $\subset$.*

Proof. Since every submodule of M is β^* equivalent to a supplement submodule in M, by [2, Theorem 3.6], M is supplemented. Since M is $\pi-$projective and supplemented, by Corollary 2.10, A is an amply supplemented lattice by the operation $\subset$. □

Corollary 2.15. *Let M be a $\pi-$projective $R-$module and A be the family of all submodules of M. If every submodule of M is β^* equivalent to a supplement submodule in M, then A is an amply g-supplemented lattice by the operation $\subset$.*

Proof. By Lemma 2.14, A is an amply supplemented lattice. Then by Lemma 2.2, A is amply g-supplemented. □

Corollary 2.16. *Let M be a projective $R-$module and A be the family of all submodules of M. If every submodule of M is β^* equivalent to a supplement submodule in M, then A is an amply supplemented lattice by the operation $\subset$.*

Proof. Clear from Lemma 2.14, since every projective module is $\pi-$projective. □

Corollary 2.17. *Let M be a projective $R-$module and A be the family of all submodules of M. If every submodule of M is β^* equivalent to a supplement submodule in M, then A is an amply g-supplemented lattice by the operation $\subset$.*

Proof. By Corollary 2.16, A is an amply supplemented lattice. Then by Lemma 2.2, A is amply g-supplemented. □

Theorem 2.18. *Let L be a lattice and $1 = a \vee b$ with $a, b \in L$. If a and b have ample g-supplements in L, then $a \wedge b$ has ample g-supplements in L.*

Proof. Let $(a \wedge b) \vee t = 1$ with $t \in L$. Since $(a \wedge b) \vee t = 1$, $a = a \wedge 1 = a \wedge ((a \wedge b) \vee t) = (a \wedge b) \vee (a \wedge t)$ and $1 = a \vee b = (a \wedge b) \vee (a \wedge t) \vee b = b \vee (a \wedge t)$. Similarly we can see that $a \vee (b \wedge t) = 1$. Since $a \vee (b \wedge t) = 1$, by hypothesis, a has a g-supplement x in L with $x \leq b \wedge t$. Here $1 = a \vee x$ and $a \wedge x \ll_g x/0$. Since $b \vee (a \wedge t) = 1$, by hypothesis, b has a g-supplement y in L with $y \leq a \wedge t$. Here $1 = b \vee y$ and $b \wedge y \ll_g y/0$. Since $1 = b \vee y$ and $y \leq a$, $a = a \wedge 1 = a \wedge (b \vee y) = (a \wedge b) \vee y$ and $1 = a \vee x = (a \wedge b) \vee x \vee y$. Since $a \wedge x \ll_g x/0$ and $b \wedge y \ll_g y/0$, by Lemma 1.1, $(a \wedge b) \wedge (x \vee y) \leq a \wedge (x \vee (b \wedge y)) = (a \wedge x) \vee (b \wedge y) \ll_g (x \vee y)/0$. Hence $x \vee y$ is a g-supplement of $a \wedge b$ in L. Moreover, $x \vee y \leq t$. Hence $a \wedge b$ has ample g-supplements in L. □

Theorem 2.19. *Let L be an amply g-supplemented lattice. Then $1/a$ is amply g-supplemented for every $a \in L$.*

Proof. Let $a \in L$ and $1 = x \vee y$ with $x, y \in 1/a$. Since L is amply g-supplemented, x has a g-supplement c in L with $c \leq y$. Since c is a g-supplement of x in L, by [8, Lemma 5], $a \vee c$ is a g-supplement of x in $1/a$. Moreover, $a \vee c \leq y$. Hence $1/a$ is amply g-supplemented, as desired. □

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