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SAKAGUCHI TYPE FUNCTION DEFINED BY $(\mathfrak{p},\mathfrak{q})$-FRACTIONAL OPERATOR USING LAGUERRE POLYNOMIALS

S. Baskaran, G. Saravanan †and B.Vanithakumari

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Abstract: An introduction of a new subclass of bi-univalent functions involving Sakaguchi type functions defined by$(\mathfrak{p},\mathfrak{q})$-fractional operators using Laguerre polynomials have been obtained. Further, the bounds for initial coefficients $|a_2|$, $|a_3|$ and Fekete Szegö inequality have been estimated.

1 Introduction and preliminaries

A function of one or more complex variables which is complex- valued is said to be analytic if it is differentiable at every point of the domain. Every normalized analytic function can be expressed as a series of the form

$$\mathfrak{f}(z) = z + \sum_{t=2}^{\infty} a_t z^t \tag{1.1}$$

in the complex variable z, that is convergent in $\mathfrak{U} = \{z : z \in \mathbb{C}, |z| < 1\}$. Let A consist of every such function. A subclass $\mathcal{S}$ of A be defined by $\mathcal{S} = \{\mathfrak{f}(z) \in A : \mathfrak{f}(z_1) = \mathfrak{f}(z_2) \Rightarrow z_1 = z_2\}$ (i.e.) $\mathcal{S}$ consists of all univalent functions.

A function $\mathfrak{f}(z) \in A$ is called bi-univalent in $\mathfrak{U}$, if $\mathfrak{f}(z) \in \mathcal{S}$ and its inverse function has an analytic continuation to $|w| < 1$. Let $\sigma = \{\mathfrak{f} \in \mathcal{S} : \mathfrak{f}$ is bi-univalent$\}$.

Though Lewin [5] introduced the class of bi-univalent functions, the passion on the bounds for the coefficients of these classes was upraised by Netanyahu, Clunie, Brannan and many others [1, 6, 10, 11, 12]. This has been a field of fascination for young researchers till date.

If, for $\mathfrak{f}_1(z)$ and $\mathfrak{f}_2(z)$ analytic in $\mathfrak{U}$, there exists a Schwarz function $\mathfrak{w}(z)$ with $\mathfrak{w}(0) = 0$ and $|\mathfrak{w}(z)| < 1$ in $\mathfrak{U}$ such that $\mathfrak{f}_1(z) = \mathfrak{f}_2(\mathfrak{w}(z))$, then we say that $\mathfrak{f}_1(z) \prec \mathfrak{f}_2(z)$.

A subclass consisting of functions satisfying the analytic criterion $Re\left(\frac{z\mathfrak{f}'(z)}{\mathfrak{f}(z)-\mathfrak{f}(-z)}\right) > \alpha$ was introduced by Sakaguchi [9] and these functions were named after him as Sakaguchi type functions [7, 8]. Sakaguchi type functions are Starlike with respect to symmetric points. Frasin [3] generalized Sakaguchi type class which had functions of the form (1.1) given by $Re\left(\frac{(\mathfrak{s}_1-\mathfrak{s}_2)z\mathfrak{f}'(z)}{\mathfrak{f}(\mathfrak{s}_1 z)-\mathfrak{f}(\mathfrak{s}_2 z)}\right) > \alpha$, $0 \le \alpha < 1, \mathfrak{s}_1, \mathfrak{s}_2 \in \mathbb{C}$ with $\mathfrak{s}_1 \neq \mathfrak{s}_2, |\mathfrak{s}_2| \le 1, \forall z \in \mathfrak{U}$.

Definition 1.1. For $\mathfrak{q}, \mathfrak{p} \in (0, 1]$ and $\mathfrak{q} < \mathfrak{p}$, the $(\mathfrak{p},\mathfrak{q})$ derivative operator $\mathfrak{D}_{\mathfrak{p},\mathfrak{q}}(\mathfrak{f}(z))$ [2] is defined as

$$\mathfrak{D}_{\mathfrak{p},\mathfrak{q}}(\mathfrak{f}(z)) = \frac{\mathfrak{f}(\mathfrak{p}z) - \mathfrak{f}(\mathfrak{q}z)}{(\mathfrak{p}-\mathfrak{q})(z)}, z \neq 0 \tag{1.2}$$

*corresponding author
†corresponding author

and $\mathfrak{D}_{\mathfrak{p},\mathfrak{q}}(\mathfrak{f}(0)) = \mathfrak{f}'(0)$ provided $\mathfrak{f}'(0)$ exists. It can be easily deduced that

$$\mathfrak{D}_{\mathfrak{p},\mathfrak{q}}(\mathfrak{f}(z)) = 1 + \sum_{t=2}^{\infty}[t]_{\mathfrak{p},\mathfrak{q}} a_t z^{t-1},$$

where $[t]_{\mathfrak{p},\mathfrak{q}} = \frac{\mathfrak{p}^t - \mathfrak{q}^t}{\mathfrak{p}-\mathfrak{q}}$, the $(\mathfrak{p},\mathfrak{q})$ bracket of t. It is also called a twin- basic number. It is to be noted that $\mathfrak{D}_{p,q}(z^t) = [t]_{\mathfrak{p},\mathfrak{q}} z^{t-1}$. Also for $\mathfrak{p} = 1$, the$(\mathfrak{p},\mathfrak{q})$ derivative operator $\mathfrak{D}_{\mathfrak{p},\mathfrak{q}}$ reduces to the $\mathfrak{q}$-derivative operator $\mathfrak{D}_{\mathfrak{q}}$.

The inverse series of (1.2) is given by

$$\begin{aligned}\mathfrak{D}_{\mathfrak{p},\mathfrak{q}}(\mathfrak{g}(w)) &= \frac{\mathfrak{g}(\mathfrak{p}w) - \mathfrak{g}(\mathfrak{q}w)}{(\mathfrak{p}-\mathfrak{q})w} \\ &= 1 - [2]_{\mathfrak{p},\mathfrak{q}} a_2 w + [3]_{\mathfrak{p},\mathfrak{q}}(2a_2^2 - a_3)w^2 \\ &\quad -[4]_{\mathfrak{p},\mathfrak{q}}(5a_2^3 - 5a_2a_3 + a_4)w^3 + \cdots .\end{aligned}$$

Consider the differential equation [4]

$$\mathfrak{x}y'' + (1 + \delta - \mathfrak{x})y' + ty = 0, \tag{1.3}$$

where $\delta + 1 > 0$, $\delta \in \mathbb{R}$ and t is non negative. The polynomial solution $y(\mathfrak{x})$ to this differential equation is said to be the generalized Laguerre polynomial or associated Laguerre polynomial and it is denoted by $\mathfrak{L}_t^{\delta}(\mathfrak{x})$. It has several applications in Mathematical physics and quantum mechanics. For example in integration of Helmholtz's equation in paraboloidal coordinates and also in theory of propagation of electromagnetic oscillations. These polynomials satisfy given recurrence relations, such as

$$\mathfrak{L}_{t+1}^{\delta}(\mathfrak{x}) = \frac{2t + 1 + \delta - \mathfrak{x}}{t+1}\mathfrak{L}_t^{\delta}(\mathfrak{x}) - \frac{t+\delta}{t+1}\mathfrak{L}_{t-1}^{\delta}(\mathfrak{x}) \quad (t \geq 1) \tag{1.4}$$

with the initial values

$$\mathfrak{L}_0^{\delta}(\mathfrak{x}) = 1, \mathfrak{L}_1^{\delta}(\mathfrak{x}) = 1 + \delta - \mathfrak{x}, \mathfrak{L}_2^{\delta}(\mathfrak{x}) = \frac{\mathfrak{x}^2}{2} - (\delta+2)\mathfrak{x} + \frac{(\delta+1)(\delta+2)}{2} \tag{1.5}$$

We obtain this equation from (1.4)

$$\mathfrak{L}_3^{\delta}(\mathfrak{x}) = \frac{-\mathfrak{x}^3}{6} + \frac{(\delta+3)}{2}\mathfrak{x}^2 - \frac{(\delta+2)(\delta+3)}{2}\mathfrak{x} + \frac{(\delta+1)(\delta+2)(\delta+3)}{6},$$

and so on.
We can see that by putting $\delta = 0$, in generalized Laguerre polynomial we get Laguerre polynomials such as

$$\mathfrak{L}_t^0(\mathfrak{x}) = \mathfrak{L}_t(\mathfrak{x}).$$

Lemma 1.2. *Let $\mathfrak{F}(\mathfrak{x}, z)$ be the generating function of the generalized Laguerre polynomial*

$$\mathfrak{F}(\mathfrak{x}, z) = \sum_{t=0}^{\infty}\mathfrak{L}_t^{\delta}(\mathfrak{x})z^t = \frac{e^{-\frac{\mathfrak{x}z}{(1-z)}}}{(1-z)^{\delta+1}}, \quad (\mathfrak{x} \in \mathbb{R}, z \in \mathfrak{U}). \tag{1.6}$$

2 Main results

Definition 2.1. A function $\mathfrak{f} \in \sigma$ is said to be in the class $\mathcal{S}_{\sigma}^{\mathfrak{pq}}(\mathfrak{x}, \delta, \mathsf{s}_1, \mathsf{s}_2)$, if the following subordination relations hold

$$\frac{(\mathsf{s}_1 - \mathsf{s}_2)z\mathfrak{D}_{\mathfrak{p},\mathfrak{q}}(\mathfrak{f}(z))}{\mathfrak{f}(\mathsf{s}_1 z) - \mathfrak{f}(\mathsf{s}_2 z)} \prec \mathfrak{F}(\mathfrak{x}, z) \tag{2.1}$$

and

$$\frac{(\mathsf{s}_1 - \mathsf{s}_2)w\mathfrak{D}_{\mathfrak{p},\mathfrak{q}}(\mathfrak{g}(w))}{\mathfrak{g}(\mathsf{s}_1 w) - \mathfrak{g}(\mathsf{s}_2 w)} \prec \mathfrak{F}(\mathfrak{x}, w) \tag{2.2}$$

where $\mathfrak{g}(w) = \mathfrak{f}^{-1}(w), \mathsf{s}_1, \mathsf{s}_2 \in \mathbb{C}$ with $\mathsf{s}_1 \neq \mathsf{s}_2, |\mathsf{s}_2| \leq 1$.

Theorem 2.2. *Let $\mathfrak{f}$ given by (1.1) be in the class $\mathcal{S}_{\sigma}^{\mathfrak{pq}}(\mathfrak{x},\delta,\mathsf{s}_1,\mathsf{s}_2)$. Then*

$$|a_2| \leq \frac{|1+\delta-\mathfrak{x}|\sqrt{|1+\delta-\mathfrak{x}|}}{\sqrt{\left|(1+\delta-\mathfrak{x})^2\mathfrak{A}-\left(\frac{\mathfrak{x}^2}{2}-(\delta+2)\mathfrak{x}+\frac{(\delta+1)(\delta+2)}{2}\right)\mathfrak{B}^2\right|}} \tag{2.3}$$

and

$$|a_3| \leq \left|\frac{1+\delta-\mathfrak{x}}{\mathfrak{C}}\right| + \frac{(1+\delta-\mathfrak{x})^2}{\mathfrak{B}^2} \tag{2.4}$$

where
$\mathfrak{A} = [3]_{pq} - [2]_{pq}(\mathsf{s}_1+\mathsf{s}_2) + \mathsf{s}_1\mathsf{s}_2,$
$\mathfrak{B} = [2]_{pq} - \mathsf{s}_1 - \mathsf{s}_2,$
$\mathfrak{C} = [3]_{pq} - \mathsf{s}_1^2 - \mathsf{s}_2^2 - \mathsf{s}_1\mathsf{s}_2.$

Proof. Let $\mathfrak{f} \in \mathcal{S}_{\sigma}^{\mathfrak{pq}}(\mathfrak{x},\delta,\mathsf{s}_1,\mathsf{s}_2)$. Then, there exist analytic functions $\phi,\psi : \mathfrak{U} \to \mathfrak{U}$ given by equation (2.1) and (2.2) such that

$$\frac{(\mathsf{s}_1-\mathsf{s}_2)z\mathfrak{D}_{\mathfrak{p},\mathfrak{q}}(\mathfrak{f}(z))}{\mathfrak{f}(\mathsf{s}_1z)-\mathfrak{f}(\mathsf{s}_2z)} = \mathfrak{F}(\mathfrak{x},\phi(z)) \tag{2.5}$$

and

$$\frac{(\mathsf{s}_1-\mathsf{s}_2)w\mathfrak{D}_{\mathfrak{p},\mathfrak{q}}(\mathfrak{g}(w))}{\mathfrak{g}(\mathsf{s}_1w)-\mathfrak{g}(\mathsf{s}_2w)} = \mathfrak{F}(\mathfrak{x},\psi(w)) \tag{2.6}$$

Define the functions $\phi(z)$ and $\psi(w)$ as

$$\phi(z) = d_1z + d_2z^2 + d_3z^3 + ..., \tag{2.7}$$

and

$$\psi(w) = e_1w + e_2w^2 + e_3w^3 + ... \tag{2.8}$$

which are analytic in $\mathfrak{U}$ with $\phi(0)$=0 , $\psi(0) = 0$ and $|\phi(z)| < 1, |\psi(w)| < 1, \quad (z,w \in \mathfrak{U})$.
We know that, if

$$|\phi(z)| = |d_1z + d_2z^2 + d_3z^3 + \cdots| < 1 \quad (z \in \mathfrak{U})$$

and

$$|\psi(w)| = |e_1w + e_2w^2 + e_3w^3 + \cdots| < 1 \quad (w \in \mathfrak{U})$$

then

$$|d_i| \leq 1, \quad |e_i| \leq 1 \quad (i = 1,2,3,...). \tag{2.9}$$

Since

$$\begin{aligned}\frac{(\mathsf{s}_1-\mathsf{s}_2)z\mathfrak{D}_{\mathfrak{p},\mathfrak{q}}(\mathfrak{f}(z))}{\mathfrak{f}(\mathsf{s}_1z)-\mathfrak{f}(\mathsf{s}_2z)} =& 1 + ([2]_{\mathfrak{pq}} - \mathsf{s}_1 - \mathsf{s}_2)\,a_2z + \left\{\left([3]_{\mathfrak{pq}} - \mathsf{s}_1{}^2 - \mathsf{s}_2{}^2 - \mathsf{s}_1\mathsf{s}_2\right)a_3\right.\\ &\left. - \left([2]_{\mathfrak{pq}}\mathsf{s}_1 + [2]_{\mathfrak{pq}}\mathsf{s}_2 - \mathsf{s}_1{}^2 - \mathsf{s}_2{}^2 - 2\mathsf{s}_1\mathsf{s}_2\right)a_2^2\right\} \times z^2 + \cdots\end{aligned} \tag{2.10}$$

$$\begin{aligned}\frac{(\mathsf{s}_1-\mathsf{s}_2)w\mathfrak{D}_{\mathfrak{p},\mathfrak{q}}(\mathfrak{g}(w))}{\mathfrak{g}(\mathsf{s}_1w)-\mathfrak{g}(\mathsf{s}_2w)} =& 1 - ([2]_{\mathfrak{pq}} - \mathsf{s}_1 - \mathsf{s}_2)\,a_2w - \{\left([3]_{\mathfrak{pq}} - \mathsf{s}_1{}^2 - \mathsf{s}_2{}^2 - \mathsf{s}_1\mathsf{s}_2\right)a_3\\ & - \left(2[3]_{\mathfrak{pq}} - \mathsf{s}_1{}^2 - \mathsf{s}_2{}^2 - [2]_{\mathfrak{pq}}\mathsf{s}_1 - [2]_{\mathfrak{pq}}\mathsf{s}_2\right)a_2^2\} \times w^2 + \cdots\end{aligned} \tag{2.11}$$

$$\frac{(\mathsf{s}_1-\mathsf{s}_2)z\mathfrak{D}_{\mathfrak{p},\mathfrak{q}}(\mathfrak{f}(z))}{\mathfrak{f}(\mathsf{s}_1z)-\mathfrak{f}(\mathsf{s}_2z)} = [\mathfrak{L}_1^{\delta}(\mathfrak{x})d_1]z + [\mathfrak{L}_1^{\delta}(\mathfrak{x})d_2 + \mathfrak{L}_2^{\delta}(\mathfrak{x})d_1^2]z^2 + \cdots \tag{2.12}$$

$$\frac{(\mathsf{s}_1-\mathsf{s}_2)w\mathfrak{D}_{\mathfrak{p},\mathfrak{q}}(\mathfrak{g}(w))}{\mathfrak{g}(\mathsf{s}_1w)-\mathfrak{g}(\mathsf{s}_2w)} = [\mathfrak{L}_1^{\delta}(\mathfrak{x})e_1]w + [\mathfrak{L}_1^{\delta}(\mathfrak{x})e_2 + \mathfrak{L}_2^{\delta}(\mathfrak{x})e_1^2]w^2 + \cdots \tag{2.13}$$

Further from equations (2.10) to (2.13), we get following equations

$$[[2]_{pq} - \mathsf{s}_1 - \mathsf{s}_2]\,a_2 = \mathfrak{L}_1^{\delta}(\mathfrak{x})d_1 \tag{2.14}$$

$$\left[[3]_{pq} - \mathsf{s_1}^2 - \mathsf{s_2}^2 - \mathsf{s_1}\mathsf{s_2}\right] a_3 - \left[[2]_{pq}\mathsf{s_1} + [2]_{pq}\mathsf{s_2} - \mathsf{s_1}^2 - \mathsf{s_2}^2 - 2\mathsf{s_1}\mathsf{s_2}\right] a_2^2 = \mathfrak{L}_1^{\delta}(\mathfrak{x})d_2 + \mathfrak{L}_2^{\delta}(\mathfrak{x})d_1^2 \tag{2.15}$$

$$-[[2]_{pq} - \mathsf{s_1} - \mathsf{s_2}]a_2 = \mathfrak{L}_1^{\delta}(\mathfrak{x})e_1 \tag{2.16}$$

$$\left[2[3]_{pq} - \mathsf{s_1}^2 - \mathsf{s_2}^2 - [2]_{pq}\mathsf{s_1} - [2]_{pq}\mathsf{s_2}\right] a_2^2 - \left[[3]_{pq} - \mathsf{s_1}^2 - \mathsf{s_2}^2 - \mathsf{s_1}\mathsf{s_2}\right] a_3 = \mathfrak{L}_1^{\delta}(\mathfrak{x})e_2 + \mathfrak{L}_2^{\delta}(\mathfrak{x})e_1^2 \tag{2.17}$$

Adding (2.14) and (2.16), we get the following equation

$$d_1 = -e_1 \tag{2.18}$$

Further squaring and adding (2.14) and (2.16), we have

$$2\left[([2]_{pq} - \mathsf{s_1} - \mathsf{s_2})^2\right] a_2^2 = [\mathfrak{L}_1^{\delta}(\mathfrak{x})]^2[d_1^2 + e_1^2] \tag{2.19}$$

Then the addition of (2.15) and (2.17), gives

$$2[[3]_{pq} - [2]_{pq}(\mathsf{s_1} + \mathsf{s_2}) + \mathsf{s_1}\mathsf{s_2}]a_2^2 = \mathfrak{L}_1^{\delta}(\mathfrak{x})(d_2 + e_2) + \mathfrak{L}_2^{\delta}(\mathfrak{x})(d_1^2 + e_1^2) \tag{2.20}$$

From the above two equations, we obtain

$$\left[2[[3]_{pq} - [2]_{pq}(\mathsf{s_1} + \mathsf{s_2}) + \mathsf{s_1}\mathsf{s_2}][\mathfrak{L}_1^{\delta}(\mathfrak{x})]^2 - 2([2]_{pq} - \mathsf{s_1} - \mathsf{s_2})^2\mathfrak{L}_2^{\delta}(\mathfrak{x})\right] a_2^2 = [\mathfrak{L}_1^{\delta}(\mathfrak{x})]^3(d_2 + e_2) \tag{2.21}$$

A small computation leads to

$$|a_2| \leq \frac{|1+\delta-\mathfrak{x}|\sqrt{|1+\delta-\mathfrak{x}|}}{\sqrt{\left|(1+\delta-\mathfrak{x})^2\mathfrak{A} - \left(\frac{\mathfrak{x}^2}{2} - (\delta+2)\mathfrak{x} + \frac{(\delta+1)(\delta+2)}{2}\right)\mathfrak{B}^2\right|}}$$

Next, in order to obtain the bound for $|a_3|$, subtracting (2.17) from (2.15) we have

$$2[[3]_{pq} - \mathsf{s}_1^2 - \mathsf{s}_2^2 - \mathsf{s_1}\mathsf{s_2}][a_3 - a_2^2] = \mathfrak{L}_1^{\delta}(\mathfrak{x})(d_2 - e_2) + \mathfrak{L}_2^{\delta}(\mathfrak{x})(d_1^2 - e_1^2) \tag{2.22}$$

Using the equations (2.18), (2.19) in (2.22), we get

$$a_3 = \frac{\mathfrak{L}_1^{\delta}(\mathfrak{x})(d_2 - e_2)}{2\mathfrak{C}} + \frac{(\mathfrak{L}_1^{\delta}(\mathfrak{x}))^2(d_1^2 + e_1^2)}{2\mathfrak{B}^2} \tag{2.23}$$

Applying equation (1.5) in the above equation and taking modulus, we have the result

$$|a_3| \leq \left|\frac{1+\delta-\mathfrak{x}}{\mathfrak{C}}\right| + \frac{(1+\delta-\mathfrak{x})^2}{\mathfrak{B}^2}$$

□

Corollary 2.3. *Let* $\mathfrak{f}$ *given by* (1.1) *be in the class* $\mathcal{S}_{\sigma}(\mathfrak{x}, \delta, \mathsf{s_1}, \mathsf{s_2})$. *Then*

$$|a_2| \leq \frac{|1+\delta-\mathfrak{x}|\sqrt{|1+\delta-\mathfrak{x}|}}{\sqrt{\left|(1+\delta-\mathfrak{x})^2\mathfrak{A}_1 - \left(\frac{\mathfrak{x}^2}{2} - (\delta+2)\mathfrak{x} + \frac{(\delta+1)(\delta+2)}{2}\right)\mathfrak{B}_1{}^2\right|}}$$

and

$$|a_3| \leq \left|\frac{1+\delta-\mathfrak{x}}{\mathfrak{C}_1}\right| + \frac{(1+\delta-\mathfrak{x})^2}{\mathfrak{B}_1{}^2}$$

where
$\mathfrak{A}_1 = 3 - 2(\mathsf{s_1} + \mathsf{s_2}) + \mathsf{s_1}\mathsf{s_2}$,
$\mathfrak{B}_1 = 2 - \mathsf{s_1} - \mathsf{s_2}$,
$\mathfrak{C}_1 = 3 - \mathsf{s}_1^2 - \mathsf{s}_2^2 - \mathsf{s_1}\mathsf{s_2}$.

Corollary 2.4. *Let* $\mathfrak{f}$ *given by* (1.1) *be in the class* $\mathcal{S}_\sigma(\mathfrak{x},\delta,1,-1)$. *Then*

$$|a_2| \leq \frac{|1+\delta-\mathfrak{x}|\sqrt{|1+\delta-\mathfrak{x}|}}{\sqrt{\left|2(1+\delta-\mathfrak{x})^2-4\left(\frac{\mathfrak{x}^2}{2}-(\delta+2)\mathfrak{x}+\frac{(\delta+1)(\delta+2)}{2}\right)\right|}}$$

and

$$|a_3| \leq \frac{|1+\delta-\mathfrak{x}|}{2} + \frac{(1+\delta-\mathfrak{x})^2}{4}$$

Corollary 2.5. *Let* $\mathfrak{f}$ *given by* (1.1) *be in the class* $\mathcal{S}_\sigma(\mathfrak{x},\delta,1,0)$. *Then*

$$|a_2| \leq \frac{|1+\delta-\mathfrak{x}|\sqrt{|1+\delta-\mathfrak{x}|}}{\sqrt{\left|(1+\delta-\mathfrak{x})^2-\left(\frac{\mathfrak{x}^2}{2}-(\delta+2)\mathfrak{x}+\frac{(\delta+1)(\delta+2)}{2}\right)\right|}}$$

and

$$|a_3| \leq \frac{|1+\delta-\mathfrak{x}|}{2} + (1+\delta-\mathfrak{x})^2$$

2.1 Fekete-Szegö Problem for the Function Class $\mathcal{S}_\sigma^{\mathfrak{p}\mathfrak{q}}(\mathfrak{x},\delta,\mathsf{s}_1,\mathsf{s}_2)$

In this section, for functions belonging to the class $\mathcal{S}_\sigma^{\mathfrak{p}\mathfrak{q}}(\mathfrak{x},\delta,\mathsf{s}_1,\mathsf{s}_2)$, we have estimated the bounds for the linear functional.

Theorem 2.6. *Let* $\mathfrak{f}\in\sigma$ *given by* (1.1) *be in the class* $\mathcal{S}_\sigma^{\mathfrak{p}\mathfrak{q}}(\mathfrak{x},\delta,\mathsf{s}_1,\mathsf{s}_2)$. *Then*

$$|a_3-\rho a_2^2| \leq \begin{cases} \frac{|1+\delta-\mathfrak{x}|}{|\mathfrak{C}|}, & if \quad 0\leq|\rho-1|\leq\left|\frac{\mathfrak{N}}{\mathfrak{C}}\right| \\ \frac{|1+\delta-\mathfrak{x}|^3|1-\rho|}{\left|(1+\delta-\mathfrak{x})^2\mathfrak{A}-\left(\frac{\mathfrak{x}^2}{2}-(\delta+2)\mathfrak{x}+\frac{(\delta+1)(\delta+2)}{2}\right)\mathfrak{B}^2\right|} & if \quad |\rho-1|\geq\left|\frac{\mathfrak{N}}{\mathfrak{C}}\right| \end{cases}$$

where
$\mathfrak{A}=[3]_{pq}-[2]_{pq}(\mathsf{s}_1+\mathsf{s}_2)+\mathsf{s}_1\mathsf{s}_2$,
$\mathfrak{B}=[2]_{pq}-\mathsf{s}_1-\mathsf{s}_2$,
$\mathfrak{C}=[3]_{pq}-\mathsf{s}_1^2-\mathsf{s}_2^2-\mathsf{s}_1\mathsf{s}_2$,
$\mathfrak{N}=\mathfrak{A}-\frac{\left(\frac{\mathfrak{x}^2}{2}-(\delta+2)\mathfrak{x}+\frac{(\delta+1)(\delta+2)}{2}\right)\mathfrak{B}^2}{(1+\delta-\mathfrak{x})^2}$.

Proof. From (2.22), for $\rho\in\mathbb{R}$, we have

$$a_3-\rho a_2^2=(1-\rho)a_2^2+\frac{(d_2-e_2)\mathfrak{L}_1^\delta(\mathfrak{x})}{2([3]_{pq}-\mathsf{s}_1^2-\mathsf{s}_2^2-\mathsf{s}_1\mathsf{s}_2)} \tag{2.24}$$

By using (2.21) in (2.24), we have

$$\begin{aligned} a_3-\rho a_2^2 &= (1-\rho)\left[\frac{(d_2+e_2)\left(\mathfrak{L}_1^\delta(\mathfrak{x})\right)^3}{2(\mathfrak{L}_1^\delta(\mathfrak{x}))^2(\mathfrak{A})-2\mathfrak{L}_2^\delta(\mathfrak{x})\mathfrak{B}^2}\right]+\frac{(d_2-e_2)\mathfrak{L}_1^\delta(\mathfrak{x})}{2([3]_{pq}-\mathsf{s}_1^2-\mathsf{s}_2^2-\mathsf{s}_1\mathsf{s}_2)} \\ &= (1+\delta-\mathfrak{x})[(\Xi(\rho,\mathfrak{x})+\tfrac{1}{2\mathfrak{C}})d_2+(\Xi(\rho,\mathfrak{x})-\tfrac{1}{2\mathfrak{C}})e_2] \end{aligned}$$

where

$$\Xi(\rho,\mathfrak{x})=\frac{(1-\rho)(1+\delta-\mathfrak{x})^2}{2(1+\delta-\mathfrak{x})^2\mathfrak{A}-2\left(\frac{\mathfrak{x}^2}{2}-(\delta+2)\mathfrak{x}+\frac{(\delta+1)(\delta+2)}{2}\right)\mathfrak{B}^2}$$

Taking modulus, we have

$$|a_3-\rho a_2^2| \leq \begin{cases} \frac{|1+\delta-\mathfrak{x}|}{|\mathfrak{C}|}, & 0\leq|\Xi(\rho,\mathfrak{x})|\leq\frac{1}{2|\mathfrak{C}|} \\ 2|1+\delta-\mathfrak{x}||\Xi(\rho,\mathfrak{x})| & |\Xi(\rho,\mathfrak{x})|\geq\frac{1}{2|\mathfrak{C}|} \end{cases}$$

□

Corollary 2.7. *Let $\mathfrak{f}$ given by (1.1) be in the class $\mathcal{S}_\sigma(\mathfrak{r}, \delta, \mathsf{s}_1, \mathsf{s}_2)$. Then*

$$|a_3 - \rho a_2^2| \leq \begin{cases} \frac{|1+\delta-\mathfrak{r}|}{|\mathfrak{C}_1|}, & if \quad 0 \leq |\rho - 1| \leq \left|\frac{\mathfrak{N}_1}{\mathfrak{C}_1}\right| \\ \frac{|1+\delta-\mathfrak{r}|^3|1-\rho|}{\left|(1+\delta-\mathfrak{r})^2\mathfrak{A}_1 - \left(\frac{\mathfrak{r}^2}{2}-(\delta+2)\mathfrak{r}+\frac{(\delta+1)(\delta+2)}{2}\right)\mathfrak{B}_1{}^2\right|} & if \quad |\rho - 1| \geq \left|\frac{\mathfrak{N}_1}{\mathfrak{C}_1}\right| \end{cases} \tag{2.25}$$

where
$\mathfrak{A}_1 = 3 - 2(\mathsf{s}_1 + \mathsf{s}_2) + \mathsf{s}_1\mathsf{s}_2,$
$\mathfrak{B}_1 = 2 - \mathsf{s}_1 - \mathsf{s}_2,$
$\mathfrak{C}_1 = 3 - \mathsf{s}_1^2 - \mathsf{s}_2^2 - \mathsf{s}_1\mathsf{s}_2,$
$\mathfrak{N}_1 = \mathfrak{A}_1 - \frac{\left(\frac{\mathfrak{r}^2}{2}-(\delta+2)\mathfrak{r}+\frac{(\delta+1)(\delta+2)}{2}\right)\mathfrak{B}_1{}^2}{(1+\delta-\mathfrak{r})^2}.$

Corollary 2.8. *Let $\mathfrak{f}$ given by (1.1) be in the class $\mathcal{S}_\sigma(\mathfrak{r}, \delta, 1, -1)$. Then*

$$|a_3 - \rho a_2^2| \leq \begin{cases} \frac{|1+\delta-\mathfrak{r}|}{2}, \\ \qquad if \quad 0 \leq |\rho - 1| \leq \left|1 - \frac{2\left(\frac{\mathfrak{r}^2}{2}-(\delta+2)\mathfrak{r}+\frac{(\delta+1)(\delta+2)}{2}\right)}{(1+\delta-\mathfrak{r})^2}\right| \\ \frac{|1+\delta-\mathfrak{r}|^3|1-\rho|}{\left|2(1+\delta-\mathfrak{r})^2 - 4\left(\frac{\mathfrak{r}^2}{2}-(\delta+2)\mathfrak{r}+\frac{(\delta+1)(\delta+2)}{2}\right)\right|} \\ \qquad if \quad |\rho - 1| \geq \left|1 - \frac{2\left(\frac{\mathfrak{r}^2}{2}-(\delta+2)\mathfrak{r}+\frac{(\delta+1)(\delta+2)}{2}\right)}{(1+\delta-\mathfrak{r})^2}\right| \end{cases} \tag{2.26}$$

Corollary 2.9. *Let $\mathfrak{f}$ given by (1.1) be in the class $\mathcal{S}_\sigma(\mathfrak{r}, \delta, 1, 0)$. Then*

$$|a_3 - \rho a_2^2| \leq \begin{cases} \frac{|1+\delta-\mathfrak{r}|}{2}, \\ \qquad if \quad 0 \leq |\rho - 1| \leq \frac{1}{2}\left|1 - \frac{\left(\frac{\mathfrak{r}^2}{2}-(\delta+2)\mathfrak{r}+\frac{(\delta+1)(\delta+2)}{2}\right)}{(1+\delta-\mathfrak{r})^2}\right| \\ \frac{|1+\delta-\mathfrak{r}|^3|1-\rho|}{\left|(1+\delta-\mathfrak{r})^2 - \left(\frac{\mathfrak{r}^2}{2}-(\delta+2)\mathfrak{r}+\frac{(\delta+1)(\delta+2)}{2}\right)\right|} \\ \qquad if \quad |\rho - 1| \geq \frac{1}{2}\left|1 - \frac{\left(\frac{\mathfrak{r}^2}{2}-(\delta+2)\mathfrak{r}+\frac{(\delta+1)(\delta+2)}{2}\right)}{(1+\delta-\mathfrak{r})^2}\right| \end{cases} \tag{2.27}$$

3 Conclusion

We have calculated the bounds for $|a_2|$, $|a_3|$ and Fekete-Szegö inequality for functions of Sakaguchi-Type function defined by $(\mathfrak{p}, \mathfrak{q})$-fractional operator using Laguerre polynomials defined by us in this paper.

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Author information

S. Baskaran,
Department of Mathematics, Agurchand Manmull Jain college, Meenambakkam, Chennai-600114, Tamil Nadu, India.
E-mail: `sbas9991@gmail.com`

G. Saravanan corresponding author,
Department of Mathematics, Patrician College of Arts and Science, Adyar, Chennai-600020, Tamil Nadu, India.
E-mail: `gsaran825@yahoo.com`

B.Vanithakumari,
Department of Mathematics, Agurchand Manmull Jain college, Meenambakkam, Chennai-600114, Tamil Nadu, India.
E-mail: `vanithagft@gmail.com`

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