Palestine Journal of Mathematics

Vol. 11(Special Issue II)(2022) , 48–56

On a Class of Time Fractional Problems with Mixed Boundary Conditions

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Communicated by Mohamed Kharrat

MSC 2010 Classifications: Primary 35R11, 35B45; Secondary 35D35.

Keywords and phrases: Time fractional problem, a priori estimate, Strong solution.

Abstract The aim of this paper is to work out the solvability of a class of Caputo time fractional problems with mixed boundary conditions (Neumann - integral). We apply the "energy inequality" method: an a priori estimate of the solution is established, then we prove the existence which is based on the range density of the operator associated with the problem.

1 Introduction

In the rectangle Q defined by $Q = (0, 1) \times (0, T)$, we consider the fractional equation

$$\mathcal{L}v = \partial_{0t}^{\alpha} v + (-1)^m \frac{\partial^m}{\partial x^m} \left(a(x,t) \frac{\partial^m v}{\partial x^m} \right) = h(x,t) \tag{1.1}$$

where $m \geq 1$ and ∂_{0t}^{α} denotes the Caputo time fractional derivative of order $0 < \alpha < 1$ and lower bound 0, a is a continuous function satisfying $0 < c_0 \leq a \leq c_1$, subject to the initial condition

$$\ell v = v(x, 0) = \phi(x), \qquad x \in (0, 1), \tag{1.2}$$

the boundary integral conditions

$$\begin{cases} \int_0^1 v(x,t)\, dx = 0, \\ \\ \int_0^1 x v(x,t)\, dx = 0, \end{cases} \qquad t \in (0, T) \tag{1.3}$$

and the Neumann conditions

$$\begin{cases} \dfrac{\partial^k}{\partial x^k} v(0,t) = g_k(t) \\ \\ \dfrac{\partial^k}{\partial x^k} v(1,t) = \psi_k(t) \end{cases} \qquad t \in (0, T), k = \overline{1, m-1}. \tag{1.4}$$

Recently, we studied this problem with "purely integral conditions" in [4], as well as other authors (cited in the same reference) did for different values of m, or in the integer order case. Note that many instances of problems described by the equation (1.1) have been investigated (see [5], [6] and [8]), but only few of them were in the fractional order case. Here, we apply the "energy inequality" method which is a traditional functional analysis method to show its well posedness. The proposed problem contains the non-homogeneous Neumann conditions (1.4), so to work out the solvability of the problem

- First, we start by the homogenization of our problem.
- Secondly, we present some preliminaries: definitions, functional spaces and other key tools in Section 3.
- In Section 4, we establish an a priori estimate of the problem's strong solution to ascertain its uniqueness and data dependance in case of existence.

- Finally, we prove the existence in Section 5 relying on the density of the generated operator's range, and we finish by giving an example, where m=2, to illustrate the usefulness of the obtained results.

2 Problem setting

We show that we can get an equivalent problem to ours (1.1)-(1.4), with homogeneous Neumann conditions. To do so, we consider $\{p_1, ..., p_{m-1}, q_1, ..., q_{m-1}\}$ a set of $2m-2$ polynomials of degree $2m$ satisfying

$$\int_0^1 p_k(x)\,dx = \int_0^1 x p_k(x)\,dx = 0, \qquad k = \overline{1, m-1},$$

$$\int_0^1 q_k(x)\,dx = \int_0^1 x q_k(x)\,dx = 0, \qquad k = \overline{1, m-1},$$

and for $1 \le k, j \le m-1$

$$\begin{cases} \frac{\partial^k}{\partial x^k} p_j(0) = \delta_{k,j} \\ \frac{\partial^k}{\partial x^k} p_j(1) = 0 \\ \frac{\partial^k}{\partial x^k} q_j(1) = \delta_{k,j} \\ \frac{\partial^k}{\partial x^k} q_j(0) = 0 \end{cases}.$$

So by setting $u = v - w$ where

$$w(x,t) = \sum_{j=1}^{m-1} p_j(x)\, g_j(t) + \sum_{j=1}^{m-1} q_j(x)\, \psi_j(t)$$

the problem (1.1)-(1.4) is equivalent to the following

$$\mathcal{L}u = \partial_{0t}^{\alpha} u + (-1)^m \frac{\partial^m}{\partial x^m}\left(a(x,t)\frac{\partial^m u}{\partial x^m}\right) = f(x,t), \tag{2.1}$$

subject to the initial condition

$$\ell u = u(x,0) = \varphi(x), x \in (0,1), \tag{2.2}$$

the integral conditions

$$\begin{cases} \int_0^1 u(x,t)\,dx = 0, \\ \int_0^1 x u(x,t)\,dx = 0, \end{cases}, \tag{2.3}$$

and the **homogeneous** Newman conditions

$$\begin{cases} \frac{\partial^k}{\partial x^k} u(0,t) = 0 \\ \frac{\partial^k}{\partial x^k} u(1,t) = 0 \end{cases} \quad k = \overline{1, m-1}, \tag{2.4}$$

where

$$f = h - \partial_{0t}^{\alpha} w - (-1)^m \frac{\partial^m}{\partial x^m}\left(a \frac{\partial^m w}{\partial x^m}\right),$$

and

$$\varphi = \phi - \ell w.$$

In the next section, we give some necessary definitions and tools.

3 Preliminaries

- The fractional derivative ∂_{0t}^{α} is defined for a differentiable function v by

$$\partial_{0t}^{\alpha} v(t) = I^{1-\alpha} v'(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{v'(\tau)}{(t-\tau)^{\alpha}} d\tau,$$

$$= \frac{d}{dt} I^{2-\alpha} v'(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{v(\tau) - v(0)}{(t-\tau)^{\alpha}} d\tau \qquad t > 0,$$

where Γ is the gamma function and I^{α} is the Riemann-Liouville integral operator defined for $0 < \alpha < 1$ by

$$I^{\alpha} v(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{v(\tau)}{(t-\tau)^{1-\alpha}} d\tau,$$

for more about fractional calculus see [7].

- $L^2(0,T)$: the space of measurable square-integrable functions on $(0,T)$.
- $B_2^k(0,1) = \{u / \Im_x^k u \in L^2(0,1)\}$ for $k \geq 1$, $B_2^0(0,1) := L^2(0,1)$ where $\Im_x^0 u = u$ and for $k \geq 1$

$$\Im_x^k u(x,t) = \frac{1}{(k-1)!} \int_0^x \frac{u(\xi,t)}{(x-\xi)^{1-k}} d\xi.$$

The scalar product in $B_2^k(0,1)$ is defined by

$$(u,v)_{B_2^k(0,1)} = \int_0^1 \Im_x^k u \Im_x^k v dx$$

and the associated norm is

$$\|u\|_{B_2^k(0,1)} = \left\|\Im_x^k u\right\|_{L^2(0,1)}.$$

Corollary 3.1. *For $k \in \mathbb{N}$ we have*

$$\|u\|_{B_2^k(0,1)}^2 \leq \frac{1}{2^k} \|u\|_{L^2(0,1)}^2. \tag{3.1}$$

Proof. See corollary of lemma 1 in [2] for $b = 1$. □

Lemma 3.2. *for any absolutely continuous function $v(t)$ on the interval $(0,T)$, we have the inequality*

$$2v(t)\,\partial_{0t}^{\alpha} v(t) \geq \partial_{0t}^{\alpha} (v(t))^2, 0 < \alpha < 1.$$

Proof. See lemma 1 in [1]. □

Lemma 3.3. *Let a non-negative absolutely continuous function $y(t)$ satisfy the inequality*

$$\partial_{0t}^{\alpha} y(t) \leq c_1 y(t) + c_2(t), \qquad 0 < \alpha < 1,$$

for almost all t in $[0,T]$, where $c_1 > 0$ and $c_2(t)$ is an integrable non-negative function on $[0,T]$.Then

$$y(t) \le y(0)\, E_\alpha(c_1 t^\alpha) + \Gamma(\alpha)\, E_{\alpha,\alpha}(c_1 t^\alpha)\, I^\alpha c_2(t).$$

where

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)},$$

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}$$

are the Mittag-leffler functions.

Proof. See Lemma 2 in [1]. □

- Cauchy inequality with ε:

$$2AB \le \varepsilon A^2 + \frac{1}{\varepsilon} B^2 \tag{3.2}$$

where A and B are real numbers.

4 A priori estimate and consequences

To establish the existence and uniqueness of the solution of problem (2.1)-(2.4) we write it in an equivalent operator form so that it can be viewed as the solution of this operator equation

$$\mathscr{L}u = \mathcal{F}$$

where $\mathscr{L} = (\mathcal{L}, \ell)$ acts from B to H with domain of definition $D_\varphi(\mathscr{L})$ of functions $u \in L^2(0,1)$ satisfying (2.3), (2.4) and

$$\Im_x^k u, \frac{\partial^k u}{\partial x^k} \in L^2(0,1), k = \overline{1,m}.$$

B is a Banach space of functions u endowed by the finite norm

$$\|u\|_B^2 = \sup_{0 \le t \le T} I^{1-\alpha} \|u\|^2_{B_2^m(0,1)} + \|u\|^2_{L^2(Q)}$$

and H is the Hilbert space consisting of vector-valued functions $\mathcal{F} = (f, \varphi)$ with finite norm

$$\|\mathcal{F}\|_H^2 = \|f\|^2_{L^2\left(B_2^m(0,1),(0,T)\right)} + \|\varphi\|^2_{L^2(0,1)}.$$

Theorem 4.1. *There exists a positive constant c not depending on u such that*

$$\|u\|_B \le c\, \|\mathscr{L}u\|_H \tag{4.1}$$

for all u in $D_\varphi(\mathscr{L})$.

Proof. We take the scalar product in space $B_2^m(0,1)$ of equation (1.1) by multiplier $\mathsf{M}u := 2\dfrac{\partial^m}{\partial x^m}\left(\dfrac{1}{a}\Im_x^m u\right)$

$$(\mathcal{L}u, \mathsf{M}u)_{B_2^m(0,1)} = (f, \mathsf{M}u)_{B_2^m(0,1)},$$

or

$$2\left(\Im_x^m(\partial_{0t}^\alpha u) + (-1)^m a(x,t)\frac{\partial^m u}{\partial x^m}, \frac{1}{a}\Im_x^m u\right)_{L^2(0,1)} = 2\left(\Im_x^m f, \frac{1}{a}\Im_x^m u\right)_{L^2(0,1)}$$

that is

$$2\int_0^1 \partial_{0t}^\alpha(\Im_x^m u)\frac{1}{a}\Im_x^m u dx + 2(-1)^m \int_0^1 \frac{\partial^m u}{\partial x^m}\Im_x^m u dx = 2\int_0^1 \Im_x^m f \frac{1}{a}\Im_x^m u dx. \tag{4.2}$$

For the first term of left hand side of equation (4.2), lemma 3.2 using the positive boundness of the function a implies the existance of a positive constant c_2 such that

$$2\int_0^1 \partial_{0t}^{\alpha}\left(\Im_x^m u\right)\frac{1}{a}\Im_x^m u dx \geq c_2\int_0^1 \partial_{0t}^{\alpha}\left(\Im_x^m u\right)^2 dx, \tag{4.3}$$

for the second term of the left hand side, integration by parts $m-1$ times using Neumann conditions (2.4) gives

$$2\left(-1\right)^m\int_0^1 \frac{\partial^m u}{\partial x^m}\Im_x^m u dx = -2\int_0^1 \frac{\partial u}{\partial x}\Im_x u dx,$$

and one last integration using integral conditions (2.3) leads to

$$2\left(-1\right)^m\int_0^1 \frac{\partial^m u}{\partial x^m}\Im_x^m u dx = 2\int_0^1 u^2 dx. \tag{4.4}$$

For the right hand side of equation (4.2) we use the Cauchy inequality with $\varepsilon = c_2$, that is

$$2\int_0^1 \Im_x^m f\frac{1}{a}\Im_x^m u dx \leq c_2\int_0^1 \left(\Im_x^m u\right)dx + \frac{1}{c_2}\int_0^1 \left(\Im_x^m f\right)^2 dx. \tag{4.5}$$

In light of (4.3)-(4.5), we deduce from the inequality (4.2) that

$$\int_0^1 \partial_{0t}^{\alpha}\left(\Im_x^m u\right)^2 dx + \frac{2}{c_2}\int_0^1 u^2 dx \leq \int_0^1 \left(\Im_x^m u\right)^2 dx + \frac{1}{c_2^2}\int_0^1 \left(\Im_x^m f\right)^2 dx. \tag{4.6}$$

Now, in the above inequality, we drop the positive term

$$\frac{2}{c_2}\int_0^1 u^2 dx$$

and substitute t by τ, then we integrate with respect to τ from 0 to t to obtain

$$\int_0^t\int_0^1 \partial_{0t}^{\alpha}\left(\Im_x^m u\right)^2 dx d\tau \leq \int_0^t\int_0^1 \left(\Im_x^m u\right)^2 dx d\tau + \frac{1}{c_2^2}\int_0^t\int_0^1 \left(\Im_x^m f\right)^2 dx d\tau,$$

from which lemma 3.3 implies

$$\int_0^t\int_0^1 \left(\Im_x^m u\right)^2 dx d\tau \leq \frac{\Gamma\left(\alpha\right)E_{\alpha,\alpha}\left(t^{\alpha}\right)}{c_2^2}I^{\alpha+1}\int_0^1 \left(\Im_x^m f\right)^2 dx.$$

Taking into consideration

$$I^{\alpha+1}\int_0^1 \left(\Im_x^m f\right)^2 dx \leq \frac{T^{\alpha}}{\Gamma\left(\alpha+1\right)}\left\|f\right\|^2_{L^2\left(B_2^m(0,1),(0,T)\right)}$$

and

$$\int_0^t\int_0^1 \partial_{0\tau}^{\alpha}\left(\Im_x^m u\right)^2 dx d\tau = I^{1-\alpha}\left\|u\right\|^2_{B_2^m(0,1)} - \frac{t^{1-\alpha}}{\Gamma\left(2-\alpha\right)}\left\|\varphi\right\|^2_{B_2^m(0,1)},$$

one can get, from the inequality (4.6), using corollary 3.1

$$I^{1-\alpha} \|u\|^2_{B_2^m(0,1)} + \frac{2}{c_2} \int_0^t \|u\|^2_{L^2(0,1)} d\tau$$

$$\leq \gamma \left(\|f\|^2_{L^2\left(B_2^m(0,1),(0,T)\right)} + \|\varphi\|^2_{L^2(0,1)} \right)$$

where

$$\gamma = \max \left\{ \frac{\Gamma(\alpha) E_{\alpha,\alpha}(T^\alpha) T^\alpha}{c_2^2 \Gamma(1+\alpha)} + \frac{1}{c_2^2}, \frac{T^{1-\alpha}}{2^m \Gamma(2-\alpha)} \right\},$$

since the right hand side of the above inequality does not depend on t, we can take the upper bound for both sides with respect to t over $[0, T]$ and the a priori estimate 4.1 follows where

$$c = \left(\frac{\gamma}{\min\left\{1, \frac{2}{c_2}\right\}} \right)^{\frac{1}{2}}.$$

□

Proposition 4.2. *The operator $\mathscr{L}$ from B to H has a closure $\overline{\mathscr{L}}$.*

Proof. See proposition 10 in [6]. □

Consequently the a priori estimate (4.1) can be extended to cover strong solutions by passing to the limit.

Corollary 4.3. *There exists a positive constant c such that*

$$\|u\|_B \leq c \left\|\overline{\mathscr{L}} u\right\|_H \tag{4.7}$$

for all u in $D_\varphi\left(\overline{\mathscr{L}}\right)$.

The uniqueness and continuous dependence of the solution on the problem data is now guaranteed in case of existence.

5 Existence of the solution

We aim to show the range density of the operator $\mathscr{L}$ in the Hilbert space H, that is $\overline{R(\mathscr{L})} = H$. Recall that $\mathscr{L} = (\mathcal{L}, \ell)$, we use the fact that the **density** of a subset in a Hilbert space means that its **orthogonal complement** is reduced to the singleton $\{0\}$. We start by the case u belongs to $D_0(\mathscr{L})$ (i.e. $\ell u = 0$), after which follows the density in the general case $u \in D_\varphi(\mathscr{L})$, taking into consideration the fact that the operator ℓ is everywhere dense.

Theorem 5.1. *Assume for all u in $D_0(\mathscr{L})$*

$$(\mathcal{L}u, \psi)_{L^2\left(B_2^m(0,1),(0,T)\right)} = 0, \tag{5.1}$$

then ψ vanishes a.e in $L^2\left(B_2^m(0,1), (0,T)\right)$.

Proof. First consider the scalar product $(\mathcal{L}u, \psi)_{B_2^m(0,1)}$, assume a function $\theta(x,t) \in L^2(0,1)$ satisfies boundary conditions (2.3)-(2.4) and $\Im_x^k \theta, \dfrac{\partial^k \theta}{\partial x^k} \in L^2(0,1), k = \overline{1,m}$. Then we can set

$$u(x,t) = \int_0^t \theta(x,\tau) d\tau$$

so we have

$$(\mathcal{L}u,\psi)_{B_2^m(0,1)}=\left(\Im_x^m\left(\partial_{0t}^\alpha u\right)+(-1)^m a\frac{\partial^m u}{\partial x^m},\Im_x^m\psi\right)_{L^2(0,1)}$$

$$=\int_0^1\partial_{0t}^\alpha\left(\int_0^t\Im_x^m\theta(x,\tau)\,d\tau\right)\Im_x^m\psi dx+(-1)^m\int_0^1 a\Im_x^m\psi\int_0^t\frac{\partial^m}{\partial x^m}\theta(x,\tau)\,d\tau dx \tag{5.2}$$

We now express ψ in terms of θ:

$$\psi(x,t)=\frac{\partial^m}{\partial x^m}\left(\frac{1}{a}\int_0^t\Im_x^m\theta(x,\tau)\,d\tau\right)$$

to get from equation (5.2)

$$(\mathcal{L}u,\psi)_{B_2^m(0,1)}=\int_0^1\frac{1}{a}\partial_{0t}^\alpha\left(\int_0^t\Im_x^m\theta(x,\tau)\,d\tau\right)\left(\int_0^t\Im_x^m\theta(x,\tau)\,d\tau\right)dx$$

$$+(-1)^m\int_0^1\int_0^t\Im_x^m\theta(x,\tau)\,d\tau\int_0^t\frac{\partial^m}{\partial x^m}\theta(x,\tau)\,d\tau dx. \tag{5.3}$$

From lemma 3.2 we have

$$\int_0^1\frac{1}{a}\partial_{0t}^\alpha\left(\int_0^t\Im_x^m\theta(x,\tau)\,d\tau\right)\left(\int_0^t\Im_x^m\theta(x,\tau)\,d\tau\right)dx\geq\frac{c_2}{2}\int_0^1\partial_{0t}^\alpha\left(\int_0^t\Im_x^m\theta(x,\tau)\,d\tau\right)^2dx,$$

and using boundary conditions (1.3), (2.4) one can get

$$(-1)^m\int_0^1\int_0^t\Im_x^m\theta(x,\tau)\,d\tau\int_0^t\frac{\partial^m}{\partial x^m}\theta(x,\tau)\,d\tau dx=\int_0^1\left(\int_0^t\theta(x,\tau)\,d\tau\right)^2dx.$$

Hence, in light of the last two relations, the substitution of t by τ in equation (5.3) then integrating with respect to τ over $[0,t]$ yields

$$\frac{c_2}{2}I^{1-\alpha}\left\|\left(\int_0^t\Im_x^m\theta(x,\tau)\,d\tau\right)\right\|_{L^2(0,1)}^2+\left\|\left(\int_0^t\theta(x,\tau)\,d\tau\right)\right\|_{L^2(Q)}^2\leq 0,$$

by taking the upper bound ovet $[0,T]$ for both sides of the above inequality yields

$$\gamma\left(\sup_{0\leq t\leq T}I^{1-\alpha}\left\|\int_0^t\Im_x^m\theta(x,\tau)\,d\tau\right\|_{L^2(0,1)}^2+\left\|\left(\int_0^t\theta(x,\tau)\,d\tau\right)\right\|_{L^2(Q)}^2\right)\leq 0,$$

where $\gamma=\min\left\{1,\frac{c_2}{2}\right\}$. Consequently, $\theta=0$ and $\psi=0$ a.e in $L^2\left(B_2^m(0,1),(0,T)\right)$. □

Now consider the general case. Let $\mathscr{L}_0$ denotes the operator $(\mathcal{L}_0,0)$, If we use the fact that $\mathscr{L}-\mathscr{L}_0=(\mathcal{L}-\mathcal{L}_0,\ell)$ maps continuously B into H, we conclude that $R(\mathscr{L})$ is dense in H by means of the method of continuation along the parameter (see [3]).

Example 5.2 (m=2). Starting with the homogenization of the problem (1.1)-(1.4), we take

$$p(x)=x^4-\frac{7}{6}x^3-\frac{3}{4}x^2+x-\frac{19}{120},q(x)=x^4-\frac{7}{6}x^3+\frac{1}{4}x^2+\frac{1}{120}$$

and

$$w(x,t) = p(x)\,g(t) + q(x)\,\psi(t)$$

to ensure that the problem is equivalent to the homogeneous one (2.1)-(2.4).

Now, let

$$b_1 = 2880\ln 2 - 2020,\ b_2 = 3066 - 4320\ln 2,\ b_3 = 816\ln 2 - 469,$$

$$\omega(x) = b_1x^3 + b_2x^2 + b_3x - 96\left(x + \frac{1}{x+1}\right),$$

$$a(x,t) = t - \frac{x^2}{2} + 1,$$

$$\varphi(x) = x^4 - 2x^3 + x^2 - \frac{1}{30}$$

and

$$f(x,t) = \frac{\omega(x)}{\Gamma(2-\alpha)}t^{-\alpha} - \frac{2}{(x+1)^5}\left(\begin{array}{c} (-73 + 324t + 5tb_2 + 9tb_1)\,x \\ +(-216t - 164 + 45tb_1 + 10tb_2)\,x^2 \\ +(-120t - 110 + 90tb_1 + 10tb_2)\,x^3 \\ +(-60t + 125 + 90tb_1 + 5tb_2)\,x^4 \\ +(259 - 12t + 45tb_1 + tb_2)\,x^5 \\ +(162 + 9tb_1)\,x^6 \\ +36x^7 + 1044t + 1152t^2 + tb_2 - 11 \end{array}\right).$$

Obviously, the function a is positively bounded, f is in $L^2\left(B_2^m(0,1),(0,T)\right)$ for any positive value of T. The reader can check by an elementary calculation that the function u given by

$$u(x,t) = \varphi(x) + t\omega(x)$$

satisfies the equation (2.1) and fulfills both initial and boundary conditions :

- Initial condition (2.2)

$$u(x,0) = \varphi(x),$$

- Integral conditions (2.3) and Neumann conditions (2.4).

Thus, it is the desired **unique** solution.

6 Conclusion

In this paper, we proved the existence and uniqueness of the strong solution of a class of Caputo time fractional problems with mixed boundary conditions of type Integral-Neumann. We used techniques related to the homogenization of the problem suggested, and worked out the solvability relying on the “energy inequality” method. Thus, this contribution will develop the functional anlysis methods, whether in integer order case or fractional order case. In addition of the example given above, which we are working on its numerical solution using the Finite-Difference method, we are looking for the solution of the same class of problems using the Riemann-Liouville fractional derivative and/or with other types of nonlocal boundary conditions.

Competing interests

The authors declare that they have no competing interests.

Authors’ contributions

All authors discussed the results and contributed to the final manuscript.

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Received: June 10, 2021.

Accepted: September 14, 2021.