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GENERALIZATION OF SOME PROPERTIES OF IDEAL OPERATORS TO LIPSCHITZ SITUATION

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Abstract In trying to answer the sixth question posed by J. D. Farmer and W. B. Johnson in thier famous paper [*Proc. Amer. Math. Soc.* **137**, 2989-2995 (2009)], we studied some composition theorems conrcening the classes of strictly Lipschitz p-summing operators, strongly Lipschitz $p-$integral operators and strongly $p-$nuclear operators, where we generalized some results from the linear case to the Lipschtiz situation.

1 Introduction

The work is situated within the framework of the non-linear operators theory. In 1983, Pietsch was the first one to put the idea generalizing the theory of ideals of linear operators to the multi-linear (and polynomial), defined the class of ideal operators and proved some properties.

On the other hand, J. D. Farmer and W. B. Johnson in [6] introduced the concept of Lipschitz $p-$summing operators and the concept of Lipschitz $p-$integral operators. They proved that this is a true extension of the linear concept, and obtained the nonlinear counterpart of the theories of linear, which has attracted the attention of various authors may be seen, for instance [1], [2], [3] [9], where presented many works related to the Lipschitz operators ideal.

In the present paper, we study the non linear case, where we have reported results about ideal operators to have analogous for Lipschitz operators ideal.

In the linear case setting, there are other composition formulas for $p-$summing, $p-$nuclear and $p-$integral operators proved by [5], [10], which states that whenever $1 \leq p, q < \infty$ satisfy $\frac{1}{r} = min\{1, \frac{1}{p} + \frac{1}{q}\}$ and if we take a $p-$summing operator and a $q-$integral and combine them via composition, we obtain a $r-$integral. Moreover, the composition of a $q-$summing operator followed by a $p-$nuclear operator is $r-$nuclear.

In this paper, we can also prove that Lipschitz operators analogues hold the compostion. Precisely, we have

$$s\iota_r^L(S \circ T) \leq s\iota_p^L(S).s\pi_q^L(T), \text{ and } s\iota_r^L(S \circ T) \leq s\pi_p^L(S).s\iota_q^L(T),$$

$$s\nu_r^L(S \circ T) \leq s\nu_p^L(S).s\pi_q^L(T), \text{ and } s\nu_r^L(S \circ T) \leq s\pi_p^L(S).s\nu_q^L(T).$$

2 Preliminaries

We recall some definitions and properties, that we need in the paper. We refer to [2], [9], [11] for more details about the following notions. We use the notation $\text{Lip}(X, Y)$ for the space of all Lipschitz operator T from a metric space X into a Banach space Y under the Lipschitz norm $\text{Lip}(.)$, where $\text{Lip}(T)$ is the infimum of all constants $C \geq 0$ such that

$$\|T(x) - T(x')\| \leq C d_X(x, x') \text{ for all } x,\ x' \in X.$$

The notation $\mathrm{Lip}_0(X,Y)$ stands for the class of all Lipschitz operators from pointed metric space by designating a special point 0 into Banach space. For $Y=\mathbb{K}$, we write $\mathrm{Lip}_0(X,\mathbb{K}) = \mathrm{Lip}_0(X) = X^\sharp$.

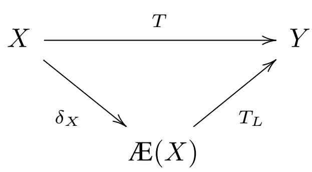
The following definitions (the space of molecules and Arens-Eells spaces) are adapted from those given in [11]. A molecule on X is a scalar valued function m on X with finite support that satisfies

$$\sum_{x\in X} m(x) = 0.$$

We denote by $\mathcal{M}(X)$ the linear space of all molecules on X. For $x, x' \in X$ the molecule $m_{xx'}$ is defined by $m_{xx'} = \chi_{\{x\}} - \chi_{\{x'\}}$, where χ_A is the characteristic function of the set A. For $m \in \mathcal{M}(X)$, we can write $m = \sum\limits_{i=1}^{n} \lambda_i m_{x_i x_i'}$, for some suitable scalars λ_i, and we write

$$\|m\|_{\mathcal{M}(X)} = \inf\left\{ \sum_{i=1}^{n} |\lambda_i| d_X(x_i, x_i'),\ m = \sum_{i=1}^{n} \lambda_i m_{x_i x_i'} \right\},$$

where the infimum is taken over all representations of the molecule m. Denote by $Æ(X)$ the completion of the normed space $(\mathcal{M}(X), \|.\|_{\mathcal{M}(X)})$. The map $\delta_X : X \to Æ(X)$ defined by $\delta_X(x) = m_{x0}$ isometrically embeds X in $Æ(X)$. Given $T \in \mathrm{Lip}_0(X,E)$, there exists a unique linear operator $T_L : Æ(X) \to Y$ such that the following diagram commutes:



The operator T_L is referred to the linearization of T (see [11, Theorem 2.2.4 (b)]). The correspondence $T \leftrightarrow T_L$ establishes an isomorphism between the vector spaces $\mathrm{Lip}_0(X,Y)$ and $\mathcal{L}(Æ(X), Y)$.
Let X be a pointed metric space, we use the notation for the Lipschitz cross-norm defined on $X \boxtimes Y$ and if X be Banach space tensor norms for norms defined $X \otimes Y$.

we are going to present some concepts about the Chevet-Saphar norms, we refer the reader to [11] for more information on these spaces.
We write d_p^L, the corresponding norm of the Chevet-Saphar norms, as follows: for every $u \in X \boxtimes Y$ we have

$$d_p^L(u) = \inf\left\{ \|m_i\|_{\ell^n_{wp^*}(Æ(X))} \|(e_i)_i\|_{\ell^n_p(Y)} \right\}$$

where the infimum is taken over all representations of u of the form $u = \sum\limits_{i=1}^{n} m_i \otimes e_i \in Æ(X) \otimes Y$.

K. Saadi [9] introduced natural notion of strictly Lipschitz $p-$summing. Let $1 \leq p \leq \infty$. A Lipschitz operator $T : X \to Y$ from a pointed metric space X into a Banach space Y is said to be strictly Lipschitz $p-$summing if there exists a positive constant C such that for all $(x_i)_{i=1}^n$, $(y_i)_{i=1}^n$ in X and $s_i^* \in Y^*(1 \leq i \leq n)$ we have

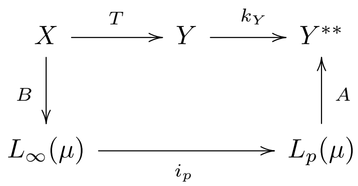
$$\left| \sum_{i=1}^{n} \langle Tx_i - Ty_i, s_i^* \rangle \right| \leq C d_p^L(u), \tag{2.1}$$

where $u = \sum\limits_{i=1}^{n} \delta_{(x_i,y_i)} \boxtimes s_i^*$, so that $\delta_{(x,y)} = m_{xy}$. The space of all strictly $p-$summing operators denoted by $\Pi_p^{SL}(X,Y)$ with its norm

$$s\pi_p^L(T) = \inf\{C > 0,\ C \text{ verifying } (2.1)\}.$$

Chen and Zheng introduced the notion of strongly Lipschitz $p-$integral operators and the notion of strongly Lipschitz $p-$nuclear operators [3].

Let us recall that if X and Y be Banach spaces, a Lipschitz operator $T : X \to Y$ is said to be strongly Lipschitz $p-$integral $(1 \le p \le \infty)$ if there are a probability measure space (Ω, Σ, μ) and a Lipschitz operator $A : L_p(\mu) \to Y^{**}$ and a bounded linear operator $B : X \to L_\infty(\mu)$ giving rise to the following commutative diagram



We define the strongly Lipschitz $p-$integral norm

$$s\iota_p^L(T) = \inf \mathrm{Lip}(A) \cdot \|B\|,$$

into a Banach space Y is strongly Lipschitz $p-$nuclear if T can be written in the form

$$T = \sum_n f_n \otimes y_n$$

where $(f_n)_n \subset X^\sharp$ and $(y_n)_n \subset Y$ satisfy $N_p^L((f_n), (y_n)_n) < \infty$, such that

$$N_p^L\left((f_n)_n, (y_n)_n\right) = \left(\mathrm{Lip}(f_n)\right)\left(\sup_n \|y_n\|\right), \qquad p = 1$$

$$N_p^L\left((f_n)_n, (y_n)_n\right) = \left(\sum_n \mathrm{Lip}(f_n)^p\right)^{\frac{1}{p}} \sup_{y^* \in B_{Y^*}} \left(\sum_n | < y^*, y_n > |^{p^*}\right)^{\frac{1}{p^*}}, \qquad 1 < p < \infty$$

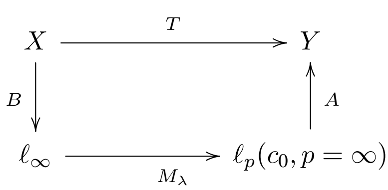
$$N_p^L\left((f_n)_n, (y_n)_n\right) = \left(\sup_n \mathrm{Lip}(f_n)\right) \sup_{y^* \in B_{Y^*}} \left(\sum_n | < y^*, y_n > |\right), \qquad \lim_n \mathrm{Lip}(f_n) = 0,\ p = \infty$$

By $\mathcal{N}_p^L(X, Y)$ we denote the space of all strongly Lipschitz $p-$nuclear operators from X into Y endowed with the strongly Lipschitz $p-$nuclear norm

$$s\nu_p^L(T) = \inf N_p^L((f_n)_n, (y_n)_n),$$

the infimum being taken over all such representations as above.

It is well-known (Theorem of factorization) that $T : X \to Y$ is strongly Lipschitz $p-$nuclear $(1 \le p \le \infty)$ if and only if T has a factorization $T = A \circ M_\lambda \circ B$ such that the following diagram commutes:



where B is a Lipschitz operator from X into ℓ_∞ with $B(0) = 0$, $M_\lambda \in \mathcal{L}(\ell_\infty, \ell_p)(\mathcal{L}(\ell_\infty, c_0),\ p = \infty)$ a diagonal operator and $A \in \mathcal{L}(\ell_p, Y)\ (\mathcal{L}(c_0, Y),\ p = \infty)$. Moreover,

$$s\nu_p^L(T) = \inf \|A\|.\|M_\lambda\|.\mathrm{Lip}(B),$$

where the infimum is taken over all the above factorizations.

These propositions are the cornerstone of our study, which allow us to obtain several results in the Lipschitz case similar to those in the linear, we refer the reader to [9].

Proposition 2.1. *Let $1 \le p < \infty$. The Lipschitz operator $T : X \to Y$ is strictly Lipschitz $p-$summing if only if its linearization T_L is $p-$summing.*

Proposition 2.2. *Let $1 \le p < \infty$. The Lipschitz operator $T : X \to Y$ is strongly Lipschitz $p-$integral if and only if its linearization T_L is $p-$integral.*

Proposition 2.3. *Let $1 \le p < \infty$. The Lipschitz operator $T : X \to Y$ is strongly Lipschitz $p-$nuclear if and only if its linearization T_L is $p-$nuclear.*

3 Main results

Our contribution in our article was inspired by the idea of generalizing certain properties to Lipschitz operators ideal, where we invest in composition theorems.

Theorem 3.1. *Let X, Y and Z be Banach spaces, let $T \in \mathrm{Lip}_0(X,Y)$ and $S \in \mathrm{Lip}_0(Y,Z)$. Suppose that $1 \leq p,q,r < \infty$ satisfy $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$.*

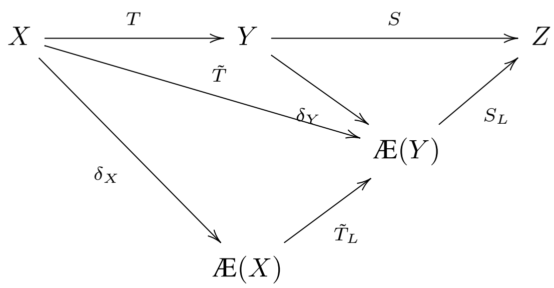
(a) If S is strongly Lipschitz $p-$integral operator and T is strictly Lipschitz $q-$summing, then $S \circ T$ is strongly Lipschitz $r-$integral operator, with

$$s\iota_r^L(S \circ T) \leq s\iota_p^L(S).s\pi_q^L(T). \tag{3.1}$$

(b) If S is strictly Lipschitz $p-$summing operator and T is strongly Lipschitz $q-$integral operator, then $S \circ T$ is strongly Lipschitz $r-$integral operator, with

$$s\iota_r^L(S \circ T) \leq s\pi_p^L(S).s\iota_q^L(T). \tag{3.2}$$

Proof. (a) Suppose that $T : X \to Y$ a strictly Lipschitz $q-$summing operator and $S : Y \to Z$ be a strongly Lipschitz $p-$integral operator. We obtain the following commutative diagram:



which shows that, $S_L : Æ(Y) \to Z$ is $p-$integral, and $\tilde{T} = \delta_Y \circ T$ is strictly Lipschitz $q-$summing by the ideal property, and note that by using [5, Theorem 5.16] and the uniqueness of linearization we find:

$$S \circ T = S_L \circ \tilde{T}_L \circ \delta_X = R \circ \delta_X,$$

where R is $r-$integral operator. Therefore, $S \circ T$ is strongly Lipschitz $r-$integral.

(b) Suppose that $S : Y \to Z$ a strictly Lipschitz $p-$summing operator and $T : X \to Y$ is strongly Lipschitz $q-$integral operator. Then $S_L : Æ(Y) \to Z$ is $q-$summing operator, and $\tilde{T} = \delta_Y \circ T$ is strongly Lipschitz $q-$integral operator by the ideal property, and so by [5, Theorem 5.16] we have:

$$S \circ T = S_L \circ \tilde{T}_L \circ \delta_X,$$

so $S_L \circ \tilde{T}_L$ is linear $r-$integral operator. Hence $S \circ T$ is strongly Lipschitz $r-$integral operator, to complete the proof. □

Theorem 3.2. *Let X be a pointed metric space and Y, Z be Banach spaces, let $T \in \mathrm{Lip}_0(X,Y)$ and $S \in \mathrm{Lip}_0(Y,Z)$. Suppose that $1 \leq p,q < \infty$ and $\frac{1}{r} = min\{1, \frac{1}{p} + \frac{1}{q}\}$.*

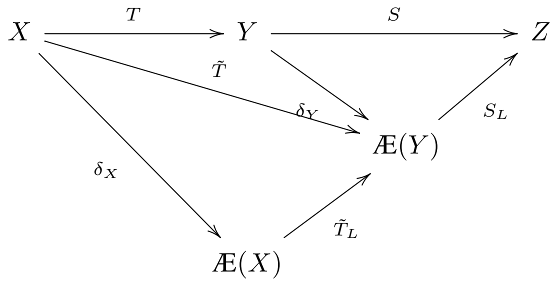
(a) If S is strongly Lipschitz $p-$nuclear operator and T is strictly Lipschitz $q-$summing operator, then $S \circ T$ is strongly Lipschitz $r-$nuclear operator, and

$$s\nu_r^L(S \circ T) \leq s\nu_p^L(S).s\pi_q^L(T). \tag{3.3}$$

(b) If S is strictly Lipschitz $p-$summing operator and T is strongly Lipschitz $q-$nuclear operator, then $S \circ T$ is strongly Lipschitz $r-$nuclear operator, and

$$s\nu_r^L(S \circ T) \leq s\pi_p^L(S).s\nu_q^L(T). \tag{3.4}$$

Proof. (a) Assume that $T : X \to Y$ a strictly Lipschitz $q-$summing and $S : Y \to Z$ be a strongly Lipschitz $p-$nuclear operator. Hence the following diagram commutes:



Now we observe that $S_L : Æ(Y) \to Z$ is $p-$nuclear, moreover $\tilde{T} = \delta_Y \circ T$ is strictly Lipschitz $q-$summing by the ideal property. we get:

$$S \circ T = S_L \circ \tilde{T}_L \circ \delta_X = R \circ \delta_X,$$

according to the theorem [10, Theorem 9.13] and the uniqueness of linearization we give: R is $r-$nuclear operator. Consequently, $S \circ T$ is strongly Lipschitz $r-$nuclear.

(b) Assume that $S : Y \to Z$ be a strictly Lipschitz $p-$summing operator and $T : X \to Y$ a strongly Lipschitz $q-$nuclear, Then $S_L : Æ(Y) \to Z$ is $p-$summing, and $\tilde{T} = \delta_Y \circ T$ is Lipschitz $q-$nuclear by factorization theorem , we have:

$$S \circ T = S_L \circ \tilde{T}_L \circ \delta_X = R \circ \delta_X,$$

where R is $r-$nuclear operator by [10, Theorem 9.13]. Then, $S \circ T$ is strongly Lipschitz $r-$nuclear, to complete the proof. □

Now, we can deduce our composition result, stating that in the linear case, the composition of a $p-$nuclear operator and a $q-$nuclear operator is $r-$nuclear operator.

Theorem 3.3. *Let $1 \leq p, q < \infty$ and $\frac{1}{r} = \min\{1, \frac{1}{p} + \frac{1}{q}\}$, if T is strongly Lipschitz $p-$nuclear operator and S a strongly Lipschitz $q-$nuclear operator. Then $T \circ S$ is strongly Lipschitz $r-$nuclear operator, and*

$$s\nu_r^L(T \circ S) \leq s\nu_p^L(T).s\nu_q^L(S).$$

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