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PRICING EUROPEAN AND AMERICAN OPTIONS UNDER FRACTIONAL MODEL

Mohamed KHARRAT

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Abstract In this paper we define and describe some models to price European and American options. We will present the both cases: the constant volatility model [6] and the stochastic volatility model [12]. In the first, using the fractional Heston model we generate a closed-form solution for the European option. The analytical solution to the fractional linear complement problem of the evaluation of American options generated by the fractional Black and Scholes model is then provided. We attempt to solve the fractional linear complementarity problem as it relates to the evaluation of American put options generated by the fractional Heston stochastic volatility model in the final section of this work. A numerical investigation is carried out to validate the theoretical results using the Adomian decomposition.. The results of this paper are published in [17], [18] and [19].

1 Introduction and Preliminaries

Pricing options is one of the most popular problems in mathematical financial literature. European and American options are extremely popular in global financial markets. Their evaluation is a challenge. The second kind allow more flexibility since it can be exercised at any time, between the current time and maturity time. Over the last few decades, several papers investigated the problem of pricing options generated by different models using many methods for instance [3], [9], [12], [13], [15],[16], [22] and [25]. The free boundary condition problem in mathematics was related to the early exercise feature inherent in American options see [5], [10], and [14] , which was more complicated. For this reason, American options have no closed form solutions. The most famous one is the Black and Scholes model [6], which is based on the idea that the underlying asset's stock price is log-normally distributed conditional on the current stock price and has a constant volatility. As compared to the case of the Black and Scholes model, where the volatility is constant, the Heston model [12] is more important since the volatility is stochastic, as the dynamics of the volatility is fundamental to elaborate strategies for hedging and for arbitrage, a model based on a constant volatility cannot explain the reality of the financial markets. So, pricing option under stochastic volatility model is then more important and required.

The fractional calculus is invested in several fields [2], [4], [7],[21], [23] and [24]. Recently, it has been integrated in the Mathematical finance field [26], [27], and especially designed to resolve the pricing option problem. For instance [11], [20], [17] and [28] which are devoted for the evaluation of the European option. Refer back to [16], [18] and [29] for the American option.

Following are some definitions related to fractional calculus, which serve as the foundation for this work. We can refer the Podlubny's book [23] as reference.

Definition 1.1. The fractional integral of order $alpha > 0$ of the Riemann-Liouville equation is defined as

$$J_{t_0}^{\alpha}y(t) = \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-\tau)^{\alpha-1}y(\tau)d\tau,$$

where $\Gamma(\alpha) = \int_0^{+\infty} e^{-t}t^{\alpha-1}dt.$

Definition 1.2. The fractional derivative of Caputo is defined as

$$D^{\alpha}_{t_0,t}x(t) = \frac{1}{\Gamma(m-\alpha)} \int_{t_0}^{t} (t-\tau)^{m-\alpha-1} \frac{d^m}{d\tau^m} y(\tau) d\tau,\ (m-1 < \alpha < m).$$

When $0 < \alpha < 1$, then the Caputo fractional derivative of order α of g reduces to

$$D^{\alpha}_{t_0,t}y(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t} (t-\tau)^{-\alpha} \frac{d}{d\tau} y(\tau) d\tau. \tag{1.1}$$

The Riemann-Liouville operator and the Caputo fractional differential operator have the following relationship.:

$$J^{\alpha}_{t_0} D^{\alpha}_{t_0,t} g(t) = D^{-\alpha}_{t_0,t} D^{\alpha}_{t_0,t} g(t) = g(t) - \sum_{k=0}^{m-1} \frac{t^k}{k!} g^k(0), \qquad m-1 < \alpha \leq m. \tag{1.2}$$

In addition to the exponential function, which is used to solve integer-order differential systems, the Mittag-Leffler function is used to solve some fractional-order differential systems.

Definition 1.3. With two parameters, the Mittag-Leffler function is defined as

$$E_{\alpha,\beta}(y) = \sum_{k=0}^{+\infty} \frac{y^k}{\Gamma(k\alpha+\beta)},$$

where $\alpha > 0$, $\beta > 0$, $z \in C$.
When $\beta = 1$, we have $E_{\alpha}(y) = E_{\alpha,1}(y)$, furthermore, $E_{1,1}(y) = e^y$.

2 The Fractional Black and Scholes Model

2.1 Pricing American put Option under Fractional Black and Scholes Model

In the financial markets the American options are the most trendy. Since the early exercise, pricing of American options was related to a problem of a free boundary condition see [5], [10], and [14], which was very complicated. For this reason, there are no closed form solutions for American options. The most famous model is the Black and Scholes model [6], which supposed the volatility as constant. Numerous works are showed so as to solve fractional differential equations, both linear and nonlinear. In this section, the Adomian decomposition method [1], [7]and [8] are used. This method is a powerful tool to find solutions for both linear or non-linear equations.

Under the hypotheses of Black and Scholes, the dynamic of the underlying asset price is given by:

$$dS_t = rS_t dt + \sigma S_t dW^S_t, \tag{2.1}$$

where S_t is the underlying asset price at time t, σ is the volatility and r is the interest rate, both are constants.

We obtain the following differential equation using the Ito formula:

$$\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma S^2 \frac{\partial^2 C}{\partial S^2} - rC = 0, \tag{2.2}$$

where C is the American put price. The time boundary conditions can be written as follows:

$$C(S_t,t) = (K - S_t, 0)^+ \qquad \textit{in the exercice case} \tag{2.3}$$

and

$$C(S_t,t) > (K - S_t, 0)^+ \qquad \textit{in the otherwise} . \tag{2.4}$$

So, pricing American put options is reduced to the following linear complementarity problem:

$$(\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma S^2 \frac{\partial^2 C}{\partial S^2} - rC)(C - (K - S_t)) = 0$$
$$\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma S^2 \frac{\partial^2 C}{\partial S^2} - rC \leq 0$$
$$C - (K - S_t) \geq 0 \quad \forall t.$$

Now we can introduce the fractional linear complementarity problem related to pricing American put options with constant volatility:

$$(\frac{\partial^\alpha C}{\partial t^\alpha} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma S^2 \frac{\partial^2 C}{\partial S^2} - rC)(C - (K - S_t)) = 0$$
$$(\frac{\partial^\alpha C}{\partial t^\alpha} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma S^2 \frac{\partial^2 C}{\partial S^2} - rC) \leq 0$$
$$C - (K - S_t) \geq 0 \quad \forall t.$$

where $0 < \alpha \leq 1$.

To determine the value of the American put option $C(S_t, V_t)$, the following nonlinear fractional differential equation must be solved:

$$D_t^\alpha C(S_t, V_t) + A[C](S_t, V_t) = 0 \qquad 0 < \alpha \leq 1 \tag{2.5}$$

in the domain $\{(S_t, V_t) | S_t \geq 0, V_t \geq 0 \; and \; t \in [0, T]\}$ with the initial value

$$C(S_0, V_0), \tag{2.6}$$

where $D_t^\alpha = \frac{\partial^\alpha}{\partial t^\alpha}$ and $A[C] = rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma S^2 \frac{\partial^2 C}{\partial S^2} - rC$.

In the case of a put option, the boundary conditions are at maturity T with an exercise price K, the payoff function is

$$(K - S_T, 0)^+. \tag{2.7}$$

Theorem 2.1. *Let $(C_t)_{t \geq 0}$ be the price American option at time t. According to the Black and Scholes hypotheses, at time l with $l < t$, the price American put option, which is the solution of:*

$$C(S_l, V_l) = max((K - S_l, 0)^+; E_\alpha(-(t - l)^\alpha A[C(S_t, V_t)])),$$

where $0 < \alpha \leq 1$, $A[C] = rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma S^2 \frac{\partial^2 C}{\partial S^2} - rC$, and E_α is the Mittag-Leffler function.

Proof. Multiplying equation (7)by the operator $D_t^{-\alpha}$ and taking into account of (2), we get

$$C(S_t, V_t) = C(S_l, V_l) + D_t^{-\alpha}(-A[C](S_t, V_t)). \tag{2.8}$$

based on the Adomian decomposition in the domain $[l, t]$, the solution has the following form:

$$C(S_t, V_t) = C(S_l, V_l) + \sum_{k=1}^{\infty} C_k(S_t, V_t). \tag{2.9}$$

By replacing (11) into (7), we have

$$C_{n+1}(S_t, V_t) = D_t^{-\alpha}(-A[C_n](S_t, V_t))$$

$$= -A[C(S_l, V_l)]^n D_t^{-\alpha}(\frac{(t - l)^{n\alpha}}{\Gamma(1 + n\alpha)}). \tag{2.10}$$

Thus, we get

$$C(S_l, V_l) = \sum_{k=0}^{\infty} (-1)^k \frac{(t - l)^{k\alpha}}{\Gamma(1 + k\alpha)} A[C(S_t, V_t)]^k$$

$$= E_\alpha(-(t - l)^\alpha A[C(S_t, V_t)]). \tag{2.11}$$

For a real and positive $alpha$, the convergence of the power series of the fractional Black and Scholes model is guaranteed. □

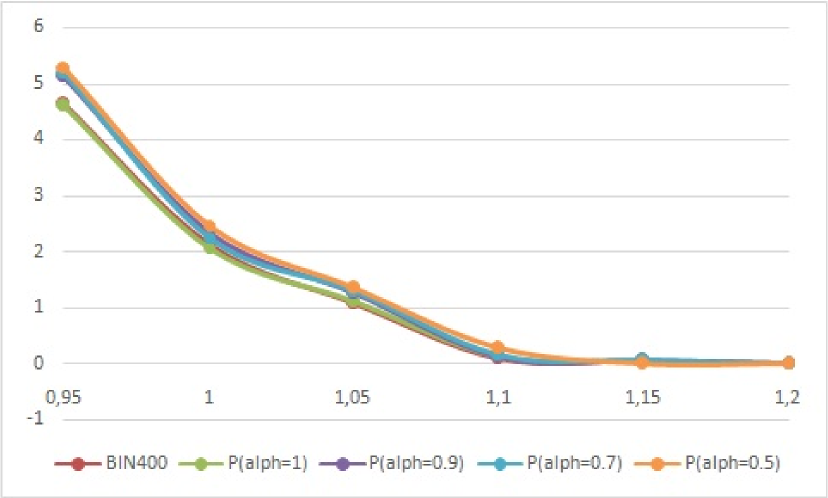

Figure 1. *Fig.1 The price of an American put option for various fractional order values as a function of moneyness, (* $\sigma = 0.2$*, r=0.05, K=100, T=1/12).*

2.2 Numerical results and simulations

In this part, we carry out the established results by presenting a numerical study of the price of an American put option for various fractional order values (see Figures 1,2 and 3).

We investigated the American put price as it relates to moneyness. In the following, we considered $\sigma = 0.2$, K=100, r=0.05. We considered three scenarios for the maturity time: the first is $1/12$, the second is $1/4$, and the third is $1/2$.

We take as reference price, the one issued from the Binomial model with 1000 steps. From the attained results, all curves have the same outlines as the one associated with the binomial model, which is consistent with option theory.

When the moneyness is located near to one, the obtained results and the binomial model differ by an insignificant margin. Otherwise, the difference in the premium of the American put option is nearly futile, as demonstrated in terms of precision, for every value of the fractional order, all results are almost the same.

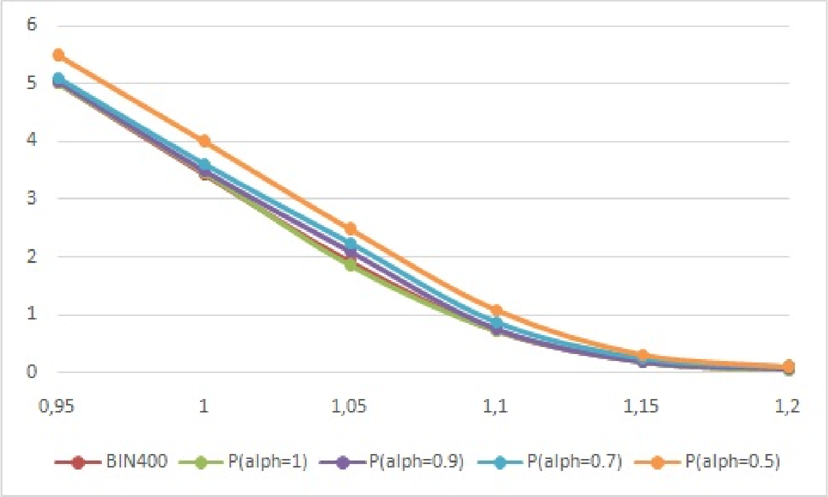

Figure 2. *Fig.5 The price of an American put option for various fractional order values as a function of moneyness, (* $\sigma = 0.2$*, r=0.05, K=100, T=1/4).*

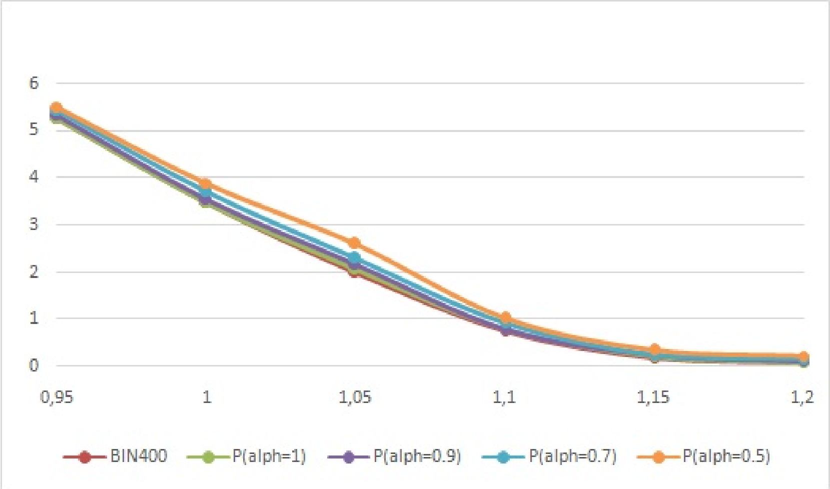

Figure 3. *Fig.5 The price of an American put option for various fractional order values as a function of moneyness, (* $\sigma = 0.2$*, r=0.05, K=100, T=1/2).*

3 Fractional Heston Model

As compared to the Black and Scholes model, the Heston model is more required since the volatility is stochastic.

Let S_t the dynamic of the asset price generated by:

$$dS_t = rS_t dt + S_t\sqrt{V_t}dW_t^S \tag{3.1}$$

and V_t follows a mean reversion and a square-root diffusion process given by:

$$dV_t = k_V(\theta_V - V_t)dt + \sigma_V\sqrt{V_t}dW_t^V, \tag{3.2}$$

where r is supposed to be constant, W_t^S and W_t^V are two standard Brownian motions that are correlated, i.e. $W_t^S = \sqrt{1-\rho^2}B_t^1 + \rho B_t^2$ and $W_t^V = B_t^2$, where B is a standard 2-dimensional Brownian motion and $\rho \in]-1,1[$. The Vt stochastic process's long-term mean, rate of mean reversion, and volatility are represented by the parameters $thetaV$, kV, and $sigmaV$, respectively.

3.1 Closed-Form European Option Solution Using Fractional Heston Model

When the volatility is stochastic, the price $C(S_t, V_t)$ of the European option is given by

$$D_t^\alpha C(S_t, V_t) + A[C](S_t, V_t) = 0, \qquad 0 < \alpha \leq 1, \tag{3.3}$$

in the domain $\{(S_t, V_t)|S_t \geq 0, V_t \geq 0 \; and \; t \in [0, T]\}$ with the initial value

$$C(S_0, V_0), \tag{3.4}$$

where $D_t^\alpha = \frac{\partial^\alpha}{\partial t^\alpha}$ and

$$A[C] = rS\frac{\partial C}{\partial S} + k(\theta - V)\frac{\partial C}{\partial V} + \frac{1}{2}VS^2\frac{\partial^2 C}{\partial S^2} + \rho\sigma VS\frac{\partial^2 C}{\partial S \partial V} - \frac{1}{2}\sigma V\frac{\partial^2 C}{\partial V^2} - rC.$$

In the case of a call option with a maturity of T and an exercise price of K, the payoff function is

$$(S_T - K, 0)^+ \tag{3.5}$$

and for the put option the payoff function is equal to

$$(K - S_T, 0)^+. \tag{3.6}$$

Theorem 3.1. *Let $(C_t)_{t\geq 0}$ be the European option price, a function of the underlying asset price and the volatility. According to the same Heston model hypotheses, the price of the European option is given by the following formula:*

$$C(S_t, V_t) = E_\alpha(-t^\alpha A[C(S_0, V_0)]),$$

where $0 < \alpha \leq 1$, $A[C] = rS\frac{\partial C}{\partial S} + k(\theta - V)\frac{\partial C}{\partial V} + \frac{1}{2}VS^2\frac{\partial^2 C}{\partial S^2} + \rho\sigma VS\frac{\partial^2 C}{\partial S\partial V} - \frac{1}{2}\sigma V\frac{\partial^2 C}{\partial V^2} - rC$ *and* E_α *is the Mittag-Leffler function*

Proof. Multiplying equation (3.3)by the operator $D_t^{-\alpha}$ and on taking into account (1.2), we get

$$C(S_t, V_t) = C(S_0, V_0) + D_t^{-\alpha}(-A[C](S_t, V_t)), \tag{3.7}$$

so, using the Adomian decomposition method we obtain

$$C(S_t, V_t) = C_0(S_t, V_t) + \sum_{k=1}^{\infty} C_k(S_t, V_t), \tag{3.8}$$

by replacing (3.8) into (3.3), we get:

$$C_{n+1}(S_t, V_t) = D_t^{-\alpha}(-A[C_n](S_t, V_t))$$

$$= -A[C(S_0, V_0)]^n D_t^{-\alpha}(\frac{t^{n\alpha}}{\Gamma(1+n\alpha)}), \tag{3.9}$$

with $C_0(S_t, V_t) = C(S_0, V_0)$, we have:

$$C(S_t, V_t) = \sum_{k=0}^{\infty}(-1)^k \frac{t^{k\alpha}}{\Gamma(1+k\alpha)} A[C(S_0, V_0)]^k$$

$$= E_\alpha(-t^\alpha A[C(S_0, V_0)]). \tag{3.10}$$

For a real and positive $alpha$, the power series of the fractional Heston model is guaranteed to converge. □

3.2 Pricing American put Option using Fractional Heston Model

Based on the Ito formula, we get the differential equation shown below:

$$\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + k(\theta - V)\frac{\partial C}{\partial V} + \frac{1}{2}VS^2\frac{\partial^2 C}{\partial S^2} + \rho\sigma VS\frac{\partial^2 C}{\partial S\partial V}$$

$$-\frac{1}{2}\sigma V\frac{\partial^2 C}{\partial V^2} - rC = 0, \tag{3.11}$$

where C represents the American put price and K represents the strike price In terms of time, the boundary conditions are as follows:

$$C(S_t, t) = (K - S_t, 0)^+ \qquad \textit{in the exercice case} \tag{3.12}$$

and

$$C(S_t, t) > (K - S_t, 0)^+ \qquad \textit{in the otherwise} . \tag{3.13}$$

In the previous section we have introduced the fractional Heston model in order to provide a European option's closed-form solution. It is now devoted to pricing American put options. So, in addition to the previous problem, We must solve the fractional linear complementarity problem shown below.:

$$[\frac{\partial^\alpha C}{\partial t^\alpha} + rS\frac{\partial C}{\partial S} + k(\theta - V)\frac{\partial C}{\partial V} + \frac{1}{2}VS^2\frac{\partial^2 C}{\partial S^2} + \rho\sigma VS\frac{\partial^2 C}{\partial S\partial V}$$

$$-\frac{1}{2}\sigma V\frac{\partial^2 C}{\partial V^2}-rC](C-(K-S_t))=0$$

$$[\frac{\partial^\alpha C}{\partial t^\alpha}+rS\frac{\partial C}{\partial S}+k(\theta-V)\frac{\partial C}{\partial V}+\frac{1}{2}VS^2\frac{\partial^2 C}{\partial S^2}+\rho\sigma VS\frac{\partial^2 C}{\partial S\partial V}$$

$$-\frac{1}{2}\sigma V\frac{\partial^2 C}{\partial V^2}-rC]\le 0$$

$$C-(K-S_t)\ge 0\quad\forall t,$$

where $0 < \alpha \le 1$.

To calculate the value of the American put price $C(St, Vt)$, under stochastic volatility, we must solve the following nonlinear fractional differential equation.

$$D_t^\alpha C(S_t,V_t)+A[C](S_t,V_t)=0 \qquad 0 < \alpha \le 1 \tag{3.14}$$

in the domain $\{(S_t,V_t)|S_t\ge 0, V_t\ge 0\; and\; t\in[0,T]\}$ with the initial value

$$C(S_0,V_0). \tag{3.15}$$

where $D_t^\alpha=\frac{\partial^\alpha}{\partial t^\alpha}$ and

$$A[C]=rS\frac{\partial C}{\partial S}+k(\theta-V)\frac{\partial C}{\partial V}+\frac{1}{2}VS^2\frac{\partial^2 C}{\partial S^2}+\rho\sigma VS\frac{\partial^2 C}{\partial S\partial V}-\frac{1}{2}\sigma V\frac{\partial^2 C}{\partial V^2}-rC.$$

In the case of a put option with a maturity of T and an exercise price of K, the payoff function is

$$(K-S_T,0)^+. \tag{3.16}$$

Theorem 3.2. *Let $(C_t)_{t\ge 0}$ be the value of the American option at time t. According to the Heston model hypotheses, at time l with $0leqlt$, the American put option price, which corresponds to the solution of the previous fractional linear complementarity problem, is equal to*

$$C(S_l,V_l)=max[(K-S_l,0)^+;E_\alpha(-(t-l)^\alpha A[C(S_t,V_t)])], \tag{3.17}$$

where $0 < \alpha \le 1$, $A[C]=rS\frac{\partial C}{\partial S}+k(\theta-V)\frac{\partial C}{\partial V}+\frac{1}{2}VS^2\frac{\partial^2 C}{\partial S^2}+\rho\sigma VS\frac{\partial^2 C}{\partial S\partial V}-\frac{1}{2}\sigma V\frac{\partial^2 C}{\partial V^2}-rC$ and E_α is the Mittag-Leffler function.

Proof. Multiplying equation (3.14)by the operator $D_t^{-\alpha}$ and taking into account (1.2), we get

$$C(S_t,V_t)=C(S_l,V_l)+D_t^{-\alpha}(-A[C](S_t,V_t)). \tag{3.18}$$

The solution is indicated as using the Adomian decomposition method in the domain $[l,t]$.

$$C(S_t,V_t)=C(S_l,V_l)+\sum_{k=1}^{\infty}C_k(S_t,V_t). \tag{3.19}$$

By substituting (3.19) into (3.14), we have

$$C_{n+1}(S_t,V_t)=D_t^{-\alpha}(-A[C_n](S_t,V_t))$$

$$=-A[C(S_l,V_l)]^nD_t^{-\alpha}(\frac{(t-l)^{n\alpha}}{\Gamma(1+n\alpha)}). \tag{3.20}$$

Thus, we get

$$C(S_l,V_l)=\sum_{k=0}^{\infty}(-1)^k\frac{(t-l)^{k\alpha}}{\Gamma(1+k\alpha)}A[C(S_t,V_t)]^k$$

$$=E_\alpha(-(t-l)^\alpha A[C(S_t,V_t)]). \tag{3.21}$$

For a real and positive $alpha$, the power series of the fractional Heston model is guaranteed to converge. □

3.3 Numerical Results and Simulations

In this section we exhibit and plot the price of the American put option under the fractional Heston model versus our reference model (see Figures 4, 5, 6, 7, 8 and 9).

We used K=100, $V0 = 0.2$, and r=0.05 as data. We considered three scenarios for time of maturity: the first is equal to $1/12$, the second is equal to $1/4$, and the third is equal to $1/2$.

The results (see Figures 4, 5, 6, 7, 8, and 9) show that all curves have the same curves as our reference model, which is consistent with option theory.

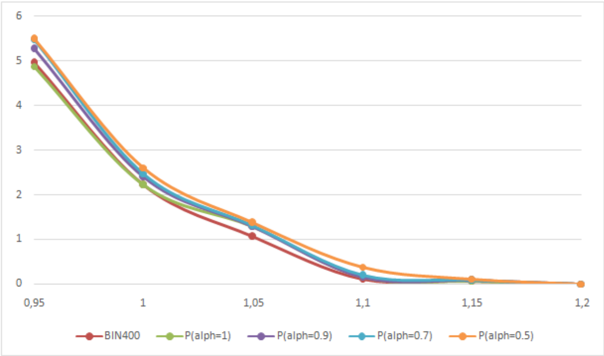

Figure 4. *The price of an American put option for various fractional order values as a function of moneyness, (K=100, $V_0 = 0.2$, r=0.05, T=1/12).*

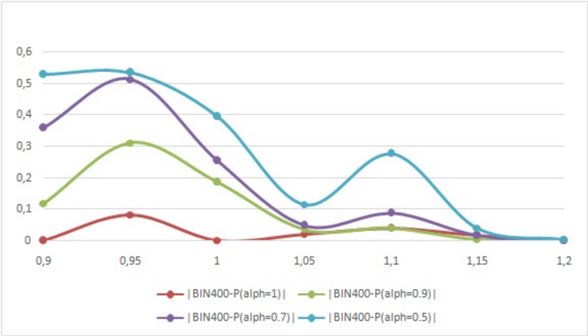

Figure 5. *The difference in the value of an American put option under the fractional Heston model and the classical binomial model (400 time steps) under stochastic volatility as a function of moneyness, (K=100, $V_0 = 0.2$, r=0.05, T=1/12).*

4 Conclusion

We provide and demonstrate the convergence of the power series related to the pricing put option problem under fractional model by utilizing the Adomian decomposition for both cases European and American option, when respectively the volatility is considered as constant and stochastic. In order to carry out the theoretical outcomes, We present numerical solutions for various fractional order values. All of the results are consistent with the theory of the American option.

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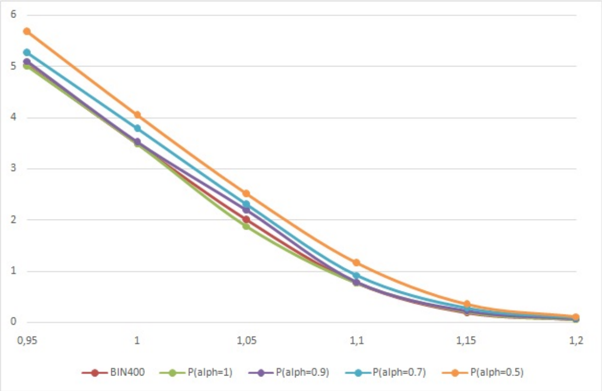

Figure 6. *Pricing American put options as a function of moneyness for different fractional model values compared to the classical binomial model (400 time-steps), (K=100,* $V_0 = 0.2$*, r=0.05, T=1/4).*

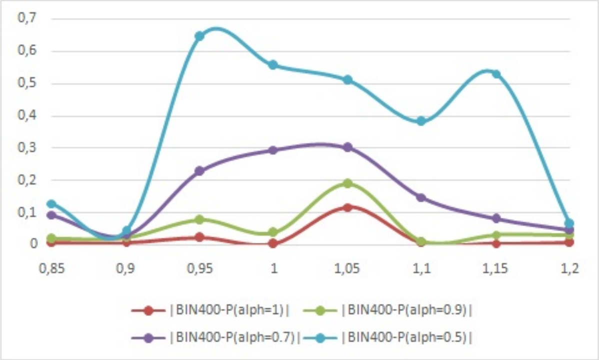

Figure 7. *The difference in the value of an American put option under the fractional Heston model and the classical binomial model (400 time steps) under stochastic volatility as a function of moneyness, (K=100,* $V_0 = 0.2$*, r=0.05, T=1/4).*

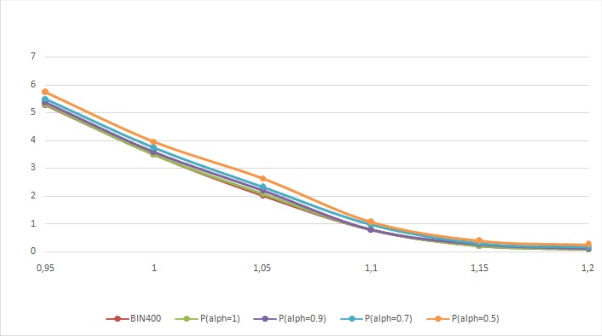

Figure 8. *Pricing American put options as a function of moneyness for various fractional model values versus the classical binomial model (400 time-steps), (K=100,* $V_0 = 0.2$*, r=0.05, T=1/2).*

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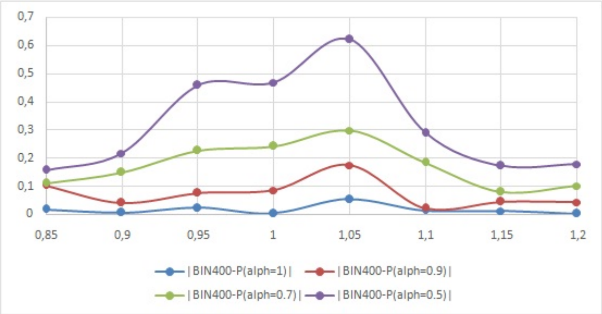

Figure 9. *As a function of moneyness, the difference in the value of an American put option under the fractional Heston model for different values of* $alpha$ *and the classical binomial model (400 time-steps) under stochastic volatility, (K=100,* $V_0 = 0.2$*, r=0.05, T=1/2).*

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Author information

Mohamed KHARRAT, Department of Mathematics, Jouf University, Saudi Arabia, and Laboratory of Probability and Statistics LR18ES28, Sfax University, Tunisia.
E-mail: mohamed.kharrat@fphm.rnu.tn

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