Palestine Journal of Mathematics

Vol. 11(Special Issue II)(2022) , 74–81

CLASSES OF ANALYTIC FUNCTIONS ASSOCIATED BY A NEW FRACTIONAL CONFORMABLE DIFFERENTIAL OPERATOR STRUCTURING BY EULER-CAUCHY EQUATIONS

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Communicated by Mohamed Kharrat

MSC 2010 Classifications: Primary 30C45; Secondary 30C55.

Keywords and phrases: Fractional calculus, conformable fractional calculus, open unit disk, analytic function, univalent function, subordination and superordination

Abstract In this note, we deal with some classes of analytic functions such as normalized, meromorphiclly multi-valent functions in the open unit disk and the puncher unit disk respectively. By using a special type of fractional differential operator, we investigate some geometric properties of these classes under the suggested operator. Moreover, as an application, we formulate a class of analytic functions which is a generalization of Euler-Cauchy equations in the open unit disk.

1 Introduction

Anderson and Ulness [1] presented a conformable differential operator (CDO) by using a notion a proportional-derivative controller for controller output μ at time t with two tuning parameters has the formula

$$\mu(t) = \lambda_p \, \Xi(t) + \lambda_d \frac{d}{dt} \Xi(t), \tag{1.1}$$

where λ_p is the proportional gain, λ_d is the derivative gain, and Ξ is the error between the process variable and the state variable. Later, Ibrahim and Jahangiri [2] proposed CDO in the open unit disk for a class of normalized functions denoting by Λ and having the series

$$g(z) = z + \sum_{n=2}^{\infty} g_n z^n, \quad z \in \mathbb{U} = \{z \in \mathbb{C} : |z| < 1\},$$

where $\mathbb{U}$ is the open unit disk, with $g(0) = g'(0) - 1 = 0$ as follows: For a fractional positive number $\nu \in [0, 1)$,

$$\begin{aligned} \mathcal{D}^0 g(z) &= g(z) \\ \mathcal{D}^\nu g(z) &= \frac{\lambda_1(\nu, z)}{\lambda_1(\nu, z) + \lambda_0(\nu, z)} \, g(z) + \frac{\lambda_0(\nu, z)}{\lambda_1(\nu, z) + \lambda_0(\nu, z)} \, (zg'(z)) \end{aligned} \tag{1.2}$$

the functions $\lambda_1, \lambda_0 : [0, 1] \times \mathbb{U} \to \mathbb{U}$ are analytic in $\mathbb{U}$ so that

$$\lambda_1(\nu, z) \neq -\lambda_0(\nu, z),$$

$$\lim_{\nu \to 0} \lambda_1(\nu, z) = 1, \quad \lim_{\nu \to 1} \lambda_1(\nu, z) = 0, \quad \lambda_1(\nu, z) \neq 0, \ \forall z \in \mathbb{U}, \ \nu \in (0, 1),$$

and

$$\lim_{\nu \to 0} \lambda_0(\nu, z) = 0, \quad \lim_{\nu \to 1} \lambda_0(\nu, z) = 1, \quad \lambda_0(\nu, z) \neq 0, \ \forall z \in \mathbb{U} \ \nu \in (0, 1).$$

It is clear that the operator (1.2) is also normalized in $\mathbb{U}$, for example,

Example 1.1. let $\lambda_1(\nu,z)=(1-\nu)z^{\nu}$, $\lambda_0(\nu,z)=\nu z^{1-\nu}$ and $g(z)=\dfrac{z}{(1-z)}$ then

$$\begin{aligned}
&\mathcal{D}^0 g(z)=\frac{z}{(1-z)}\\
&\mathcal{D}^{\nu}\left(\frac{z}{(1-z)}\right)=\frac{(1-\nu)z^{\nu}}{(1-\nu)z^{\nu}+\nu z^{1-\nu}}\left(\frac{z}{(1-z)}\right)+\frac{\nu z^{1-\nu}}{(1-\nu)z^{\nu}+\nu z^{1-\nu}}\left(z\left(\frac{z}{(1-z)}\right)'\right)\\
&=\frac{(1-\nu)z^{\nu}}{(1-\nu)z^{\nu}+\nu z^{1-\nu}}\left(\frac{z}{(1-z)}\right)+\frac{\nu z^{1-\nu}}{(1-\nu)z^{\nu}+\nu z^{1-\nu}}\left(\frac{z}{(1-z)^2}\right)\\
&=\frac{z((\nu-1)(z-1)z^{(2\nu)}+\nu z)}{(z-1)^2(\nu z-(\nu-1)z^{(2\nu)})}\\
&=((z+2z^2+3z^3+4z^4+5z^5+O(z^6))\\
&\times((\nu z+O(z^6))+z^{(2\nu)}\frac{((1-\nu)+(\nu-1)z+O(z^6))))}{((\nu z+O(z^6))-(\nu-1)z(2\nu))}.
\end{aligned}$$

Hence, the operator (1.2) is normalized in $\mathbb{U}$.

In general, we have the following example [3]

Example 1.2. Let $\phi\in\wedge$ taking the expansion formula

$$\phi(z)=z+\sum_{n=2}^{\infty}\phi_n\,z^n$$

then

$$\begin{aligned}
&\mathcal{D}^{\nu}\phi(z)\\
&=\frac{\lambda_1(\nu,z)}{\kappa_1(\nu,z)+\kappa_0(\nu,z)}\,\phi(z)+\frac{\lambda_0(\nu,z)}{\lambda_1(\nu,z)+\lambda_0(\nu,z)}\,(z\,\phi'(z))\\
&=\frac{\lambda_1(\nu,z)}{\lambda_1(\nu,z)+\lambda_0(\nu,z)}\left(z+\sum_{n=2}^{\infty}\phi_n\,z^n\right)+\frac{\lambda_0(\nu,z)}{\lambda_1(\nu,z)+\lambda_0(\nu,z)}\left(z+\sum_{n=2}^{\infty}n\phi_n\,z^n\right)\\
&=z+\sum_{n=2}^{\infty}\left(\frac{\lambda_1(\nu,z)+n\,\lambda_0(\nu,z)}{\lambda_1(\nu,z)+\lambda_0(\nu,z)}\right)\phi_n\,z^n.
\end{aligned}$$

Recently, Ibrahim and Baleanu [3, 4] employed the operator (1.2) to formulate a hybrid conformable diff-integral operator and a quantum hybrid operator respectively. In this note, we shall present some classes of analytic functions associated with CDO.

2 Meromorphically multivalent functions

Here, our discussion is based on a class of functions denoting by $\Sigma_k(\rho)$ and constructing by (see [5])

$$f(z)=z^{-\rho}+\sum_{n=k}^{\infty}a_n z^{n-\rho}, \tag{2.1}$$

which are analytic in the punctured open unit disk $U^*=\{z\in\mathbb{C}:0<|z|<1\}$.

Definition 2.1. For a function $f \in \Sigma_k(\rho)$, CDO is defined as follows:

$$\begin{aligned}
\Delta^0 f(z) &= f(z)\\
\Delta^\nu f(z) &= \frac{\lambda_1(\nu,z)}{\lambda_1(\nu,z)+\lambda_0(\nu,z)}\, f(z) + \frac{\lambda_0(\nu,z)}{\lambda_1(\nu,z)+\lambda_0(\nu,z)} \left(\frac{-z}{\rho}\right) f'(z)\\
&= z^{-\rho} + \sum_{n=k}^{\infty} a_n \left(\frac{\lambda_1(\nu,z)+((\rho-n)/\rho)\,\lambda_0(\nu,z)}{\lambda_1(\nu,z)+\lambda_0(\nu,z)}\right) z^{n-\rho}\\
\Delta^{2\nu} f(z) &= \Delta(\Delta^\nu f(z))\\
&= z^{-\rho} + \sum_{n=k}^{\infty} a_n \left(\frac{\lambda_1(\nu,z)+((\rho-n)/\rho)\,\lambda_0(\nu,z)}{\lambda_1(\nu,z)+\lambda_0(\nu,z)}\right)^2 z^{n-\rho} \qquad (2.2)\\
&\vdots\\
\Delta^{m\,\nu} f(z) &= \Delta^\nu[\Delta^{(m-1)\nu} f(z)]\\
&= z^{-\rho} + \sum_{n=k}^{\infty} a_n \left(\frac{\lambda_1(\nu,z)+((\rho-n)/\rho)\,\lambda_0(\nu,z)}{\lambda_1(\nu,z)+\lambda_0(\nu,z)}\right)^m z^{n-\rho}\\
&:= z^{-\rho} + \sum_{n=k}^{\infty} a_n\,(\Lambda_n)^m\, z^{n-\rho}.
\end{aligned}$$

$$\left(z \in U^*,\ \rho \in \mathbb{N}, \nu \in [0,1]\right)$$

where

$$\lim_{\nu\to 0} \lambda_1(\nu,z) = 1, \quad \lim_{\nu\to 1} \lambda_1(\nu,z) = 0, \quad \lambda_1(\nu,z) \neq 0,\ \forall z \in U^*,\ \nu \in (0,1),$$

and

$$\lim_{\nu\to 0} \lambda_0(\nu,z) = 0, \quad \lim_{\nu\to 1} \lambda_0(\nu,z) = 1, \quad \lambda_0(\nu,z) \neq 0,\ \forall z \in U^*\ \nu \in (0,1).$$

It is clear that

$$f \in \Sigma_k(\rho) \Rightarrow \Delta^{m\,\nu} f(z) \in \Sigma_k(\rho).$$

Definition 2.2. An analytic function φ is subordinated to an analytic function ψ, written $\varphi \prec \psi$, if occurs an analytic function h with $|h(z)| \leq |z|$ such that $\varphi = (\psi(h))$ (see [6, 7]).

We have the following geometric results

Theorem 2.3. *Define a functional*

$$\Theta(z) := (1-\sigma) z^\rho\,[\Delta^{m\,\nu} f(z)] - \left(\frac{\sigma}{\rho}\right) z^{1+\rho}[\Delta^{m\,\nu} f(z)]', \quad \sigma < 0.$$

Then

$$\Re\,(\Theta(z)) > 0 \Rightarrow |\phi_n| \leq 2\int_0^{2\pi} |e^{-in\theta}|\, d\mu(\theta),$$

where $d\mu$ is a probability measure. Moreover,

$$\Re\left(e^{i\varpi}\Theta(z)\right) > 0 \Rightarrow \Theta(z) \in \mathcal{C},$$

where $\mathcal{C}$ is the class of analytic convex in $\mathbb{U}$.

Proof. For the first part of the theorem, we suppose that

$$\Re(\Theta(z)) = \Re\left(1 + \sum_{n=1}^{\infty} \phi_n z^n\right) > 0.$$

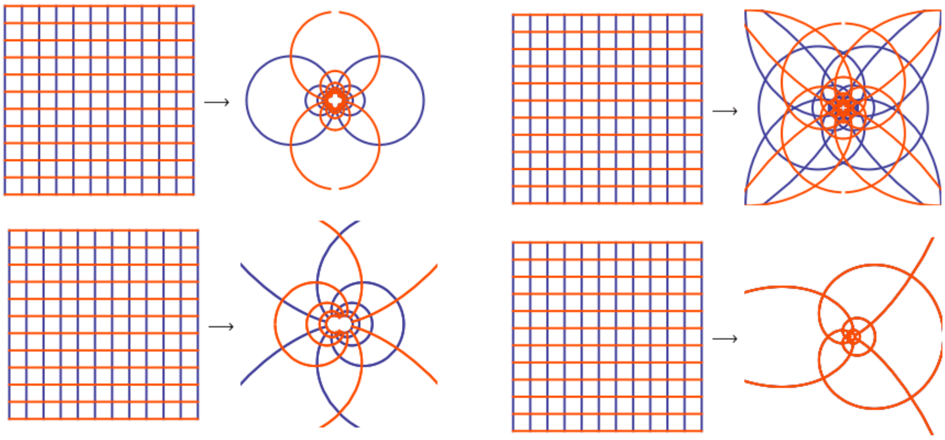

Figure 1. The plot of $f(z) = z^{-\rho}$, when $\rho = 1, 2$ (first coloumn) and $\rho = 3, 4$ (second coloumn).

Then by the Carathéodory positivist method for analytic functions, we have

$$|\phi_n| \leq 2 \int_0^{2\pi} |e^{-in\theta}| \, d\mu(\theta),$$

where $d\mu$ is a probability measure. Lastly, if

$$\Re \left(e^{i\varrho} \Theta(z) \right) > 0, \quad z \in \mathbb{U}, \ \varrho \in \mathbb{R}$$

then according to [8]-Theorem 1.6(P22) and for some real numbers ϱ, we have

$$\Theta(z) \approx \frac{Az+1}{Bz+1}, \quad z \in \mathbb{U}.$$

But $\dfrac{Az+1}{Bz+1}$ is convex in $\mathbb{U}$, then by majority concept, we obtain that $\Theta(z) \in \mathcal{C}$. □

Example 2.4. Let $\lambda_1 = \lambda_0 = 0.5z^{0.5}, \nu = 0.5$. Then for $f(z) = z^{-\rho}, where\ \rho = 1, 2, 3, 4$ and $m = 1$, we have (see Fig.1)

$$\Delta^{m\,\nu} f(z) = 0.5(\frac{1}{z} - z \times \frac{d}{dz}\frac{1}{z} = \frac{1}{z}, \quad \rho = 1;$$

$$\Delta^{m\,\nu} f(z) = 0.5(\frac{1}{z^2} - z \times \frac{d}{dz}\frac{1}{z^2}) = \frac{1.5}{z^2}, \quad \rho = 2;$$

$$\Delta^{m\,\nu} f(z) = 0.5(\frac{1}{z^3} - z \times \frac{d}{dz}\frac{1}{z^3}) = \frac{2}{z^3}, \quad \rho = 3;$$

and

$$\Delta^{m\,\nu} f(z) = 0.5(\frac{1}{z^4} - z \times \frac{d}{dz}\frac{1}{z^4}) = \frac{2.5}{z^4}, \quad \rho = 4.$$

Theorem 2.5. *Define a functional $\upsilon(z) := z^{\rho+1}\Delta^{m\,\nu} f(z), z \in \mathbb{U}$. If the subordination*

$$\upsilon(z) \prec \frac{z}{(1+z)^2}$$

is hold then $\upsilon(z) \in \mathbb{S}^*$ (the class of starlike analytic functions) and

$$\left(\int_0^z \frac{\sqrt{\upsilon(\zeta)}}{\zeta} d\zeta \right)^2 \prec \left(2 \tan^{-1} \sqrt{z}\right)^2$$

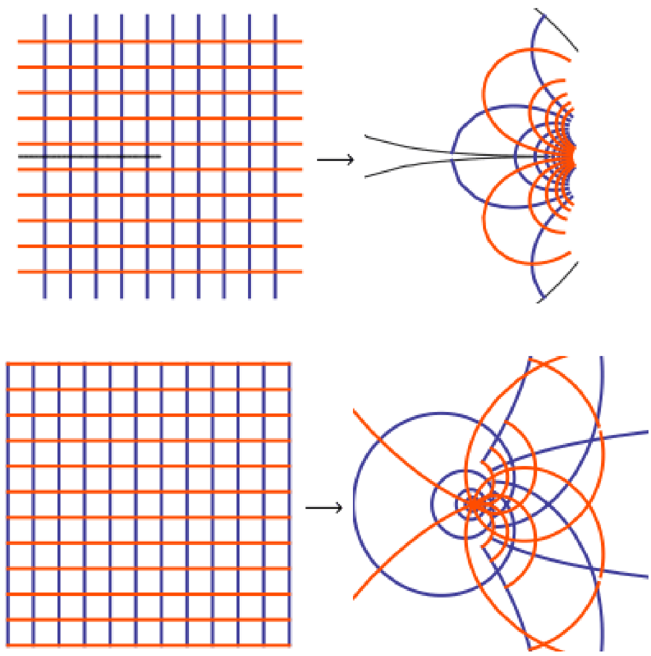

Figure 2. The plot of $\top(z)$ and $\wp(z)$ respectivelly.

such that

$$-\frac{\pi}{2} < -2\tan^{-1}\sqrt{r} \le \Re\left(\int_0^z \frac{\sqrt{\upsilon(\zeta)}}{\zeta} d\zeta\right) < 2\tan^{-1}\sqrt{r} \le \frac{\pi}{2}.$$

Proof. Let $\upsilon(z) = z^{\rho+1}\Delta^{m\,\nu} f(z), z \in \mathbb{U}$. Then

$$\upsilon(z) = z + \sum_{n=2}^{\infty} \upsilon_n z^n, \quad z \in \mathbb{U}$$

is analytic in the open unit disk, where

$$\begin{aligned}\top(z) :&= \left(2\tan^{-1}\sqrt{z}\right)^2 \\ &= 4z - 8\frac{z^2}{3} + \frac{92z^3}{45} + O(z^4).\end{aligned}$$

Since the function (see [7]-P177)

$$\begin{aligned}\wp(z) &= \frac{z}{(1+z)^2} \\ &= z - 2z^2 + 3z^3 - 4z^4 + 5z^5 + O(z^6) \in \mathbb{S}^*, \quad z \in \mathbb{U},\end{aligned}$$

then by majority concept, we have $\upsilon(z) \in \mathbb{S}^*$. The second and third assertions are verified by [7]-Corollary 3.6a.1 (see Fig.2). □

In the similar manner of Theorem 2.5, if we replace $\top(z)$ by one of the function

$$\Upsilon(z) := -(\log(1 - i\sqrt{z}) - \log(1 + i\sqrt{z}))^2, \quad z \in \mathbb{U},$$

$$\Lambda(z) := \left(2\cot^{-1}\left(\frac{1}{\sqrt{z}}\right)\right)^2, \quad z \in \mathbb{U}.$$

3 Conformable Euler-Cauchy Equations

The class of complex differential equations has concerned in many researches captivating the common arrangement

$$\lambda^{(k)}(\xi) + a_{k-1}\lambda^{(k-1)}(\xi) + ... + a_k = 0,$$

Where $\lambda(\xi)$ is an analytic function in a complex domain with non-zero coefficients. Classes of this prescription are considered extensively. Most of these studies are dedicated on the association issue and its boundary. For instance, Pommerenke considered [9] the second order; Heittokangas [10] investigated a special example of the $k-$the order, whereas Walter [11] offered a meromorphic solution for a class of complex differential equation. Later, the equation is modified utilizing fractional calculus in the open unit disk [12, 13, 14].

This section deals with a generalized class of second order differential equations type Euler-Cauchy equations (ECEs) utilizing the suggested operator CDO. The general formula of ECE is given by the structure

$$z^2\phi''(z) + az\phi'(z) + b\phi(z) = \eta(z), \quad z \in \mathbb{U}, \tag{3.1}$$

where ϕ and η are analytic in $\mathbb{U}$. Our discussion will take place on the function $\phi(z) := z^{\rho+1}\Delta^{m\,\nu}f(z), z \in \mathbb{U}$. Therefore, Eq.(3.1) can be generalized by the formula

$$z^2\left(z^{\rho+1}\Delta^{m\,\nu}f(z)\right)'' + az\left(z^{\rho+1}\Delta^{m\,\nu}f(z)\right)' + b\left(z^{\rho+1}\Delta^{m\,\nu}f(z)\right) = \eta(z), z \in \mathbb{U}. \tag{3.2}$$

Definition 3.1. Consider the normalized functions $f \in \Sigma_k(\rho)$. Then the function f is in the class $\mathcal{A}_\nu(\alpha, \eta(z))$ if it satisfies the Ma-Minda type [6] of subordination inequality

$$z^2\left(z^{\rho+1}\Delta^{m\,\nu}f(z)\right)'' + (1-\alpha)z\left(z^{\rho+1}\Delta^{m\,\nu}f(z)\right)' + \alpha\left(z^{\rho+1}\Delta^{m\,\nu}f(z)\right) \prec \eta(z).$$

To illustrate our result, we need the following lemma [7]P139-140.

Lemma 3.2. *Let $\upsilon \in \Lambda$. Then*

(a) $\upsilon(z) + \alpha z\,\upsilon'(z) \prec (1+\alpha)z + \alpha z^2 \Rightarrow \upsilon(z) \prec z$, *when* $\alpha \in (0, 1/3]$;

(b) $z\upsilon'(z)[1+\upsilon(z)] + \alpha\upsilon^2(z) \prec \xi + (1+\alpha)z^2 \Rightarrow \upsilon(z) \prec z$, *when* $|1+\alpha| \le 1/4$;

(c) $[z\upsilon'(z) - \upsilon(z)]e^{\alpha(\upsilon(z))} + e^{\upsilon(z)} \prec e^z \Rightarrow \upsilon(z) \prec z$, *when* $|\alpha - 1| \le \pi/2$;

(d) $z\upsilon'(z)(1+\alpha\upsilon(z)) + \upsilon(z) \prec 2z + \alpha z^2 \Rightarrow \upsilon(z) \prec z$, *when* $|\alpha| \le 1/2$;

(e) $z\upsilon'(z)e^{\alpha\,\upsilon(z)} + \upsilon(z) \prec z(1+\alpha z e^{\alpha z}) \Rightarrow \upsilon(z) \prec z$, *when* $|\alpha| \le 1$;

(f) $\upsilon(z) + \dfrac{z\upsilon'(z)}{1+\alpha\upsilon(z)} \prec z \Rightarrow \upsilon(z) \prec z$, *when* $|\alpha| \le 1$;

and the solution is sharp.

Theorem 3.3. *Let $f \in \Sigma_k(\rho)$. If one of the following inequalities occurs*

(a) $\alpha z^3\phi'''(z) + ((3-\alpha)\alpha + 1)\,z^2\phi''(z) + z\phi'(z) + \alpha\phi(z) \prec (1+\alpha)z + \alpha z^2$ *when* $\alpha \in (0, 1/3]$;

(b) $\left(z^3\phi'''(z) + (3-\alpha)z^2\phi''(z) + z\phi'(z)\right)[1+z^2\phi''(z)+(1-\alpha)z\phi'(z)+\alpha\phi(z)]+\alpha\left(z^2\phi''(z) + (1-\alpha)z\phi'(\right.$
$z + (1+\alpha)z^2$, *when* $|1+\alpha| \le 1/4$;

(c) $\left(z^3\phi'''(z) + (2-\alpha)z^2\phi''(z) + \alpha z\phi'(z) - \alpha\phi(z)\right)\exp\left(\alpha(z^2\phi''(z) + (1-\alpha)z\phi'(z) + \alpha\phi(z))\right)+$
$\exp\left(z^2\phi''(z) + (1-\alpha)z\phi'(z) + \alpha\phi(z)\right) \prec e^z$,
when $|\alpha - 1| \le \pi/2$;

(d) $\left(z^3\phi'''(z) + (3-\alpha)z^2\phi''(z) + z\phi'(z)\right)[1+\alpha z^2\phi''(z)+\alpha(1-\alpha)z\phi'(z)+\alpha^2\phi(z)]+\left(z^2\phi''(z) + (1-\alpha)z\phi\right.$
$2z + \alpha z^2$, *when* $|\alpha| \le 1/2$;

(e) $\left(z^3\phi'''(z) + (3-\alpha)z^2\phi''(z) + z\phi'(z)\right)\exp\left(\alpha[z^2\phi''(z) + (1-\alpha)z\phi'(z) + \alpha\phi(z)]\right)+\left(z^2\phi''(z) + (1-\alpha\right.$
$z(1+\alpha\,ze^{\alpha z})$, *when* $|a| \le 1$;

(f) $z^2\phi''(z)+(1-\alpha)z\phi'(z)+\alpha\phi(z)+\dfrac{z^3\phi'''(z)+(3-\alpha)z^2\phi''(z)+z\phi'(z)}{1+\alpha\left(z^2\phi''(z)+(1-\alpha)z\phi'(z)+\alpha\phi(z)\right)} \prec z$, *when* $|\alpha|\le 1$;

then $f\in\mathcal{A}_\nu(\alpha,z)$.

Proof. For $f\in\Sigma_k(\rho)$, let

$$\phi(z):=z^{\rho+1}\Delta^{m\,\nu}f(z),\, z\in\mathbb{U}.$$

Then $\phi\in\Lambda$. Assuming that

$$\upsilon(z):=z^2\left(z^{\rho+1}\Delta^{m\,\nu}f(z)\right)''+(1-\alpha)z\left(z^{\rho+1}\Delta^{m\,\nu}f(z)\right)'+\alpha\left(z^{\rho+1}\Delta^{m\,\nu}f(z)\right)$$

in Lemma 3.2 such that

$$\upsilon(z)=z^2\phi''(z)+(1-\alpha)z\phi'(z)+\alpha\phi(z)$$

and

$$z\upsilon'(z)=z^3\phi'''(z)+(3-\alpha)z^2\phi''(z)+z\phi'(z).$$

Consequently, by the inequalities of the theorem, we have $f\in\mathcal{A}_\nu(\alpha,z)$. □

Example 3.4. Consider the equation

$$z^2\phi''(z)+(1-\alpha)z\phi'(z)+\alpha\phi(z)=z$$

then for

- $\alpha=0.5$, the solution is formulated by

$$\phi(z)=c_1z^{0.25}\sin(0.66\log(z))+c_2z^{0.25}\cos(0.66\log(z))$$
$$+z\sin^2(0.66\log(z))+z\cos^2(0.66\log(z));$$

- $\alpha=0.25$ the solution is given by

$$\phi(z)=c_1z^{0.125}\sin(0.484123\log(z))+c_2z^{0.125}\cos(0.484123\log(z))+z.$$

4 Conclusion

From above investigation, we illustrated some geometric presentations of a conformable fractional operator in the open unit disk. The operator is suggested for a special type of analytic functions called meromorphically multivalent function. We show that under some conditions, the suggested operator is convex and starlike.

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Received: June 16, 2021.

Accepted: October 22, 2021.