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Quadratic BSDEs with two reflecting barriers and a square integrable terminal value

Roubi Abdallah, Labed Boubakeur and Khaled Bahlali

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Abstract We consider a backward stochastic differential equation (BSDE) with two reflecting barriers which generator $H(t,\omega,y,z)$ has a quadratic growth in its z-variable and a square integrable terminal value ξ. The solutions is constrained to stay between two time continuous processes L and U (called the barriers). We establish the existence of solutions when $H(t,\omega,y,z) := a + b|y| + c|z| + f(y)|z|^2$. The uniqueness and the comparison of solutions are also established when the generator is of the form $f(y)|z|^2$. The main tools are Krylov's estimate and Itô-Krylov's formula, which are proved here, for the solutions of backward stochastic differential equations with two reflecting barriers.

1 Introduction

Let $(B_t)_{t\geq 0}$ be a standard d-dimensional Brownian motion, defined on a probability space $(\Omega,\mathcal{F},\mathbb{P})$, let $\{\mathcal{F}_t\}_{0\leq t\leq T}$ be the natural filtration of B_t, where $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets of $\mathcal{F}$. On the other hand, for $p>0$, we denote:

$\mathbb{L}^p_{loc}(\mathbb{R})$:= the space of (classes) of functions u defined on $\mathbb{R}$ which are p-integrable on bounded set of $\mathbb{R}$.

$\mathcal{W}^2_{p,loc}$:= the Sobolev space of (classes) of functions u defined on $\mathbb{R}$ such that both u and its generalized derivatives u' and u'' belong to $\mathbb{L}^p_{loc}(\mathbb{R})$.

$\mathcal{C}$:= the space of continuous and $\mathcal{F}_t$-adapted processes.

$\mathcal{S}^2$:= the space of continuous,$\mathcal{F}_t$-adapted processes φ such that $\mathbb{E}\sup_{0\leq t\leq T}|\varphi_t|^2 < +\infty$.

$\mathbb{L}^2$:= the space of $\mathcal{F}_T-$measurable random variable ξ s.t. satisfying $\mathbb{E}|\xi|^2 < +\infty$.

$\mathcal{M}^2$:= the space of $\mathcal{F}_t$-adapted processes φ satisfying $\mathbb{E}\int_0^T|\varphi_t|^2\,dt < +\infty$.

$\mathcal{L}^2$:= the space of $\mathcal{F}_t$-adapted processes φ satisfyin $\int_0^T|\varphi_t|^2\,dt < +\infty$.

$\mathcal{K}$:= the space of $\mathcal{P}$-measurable continuous nondecreasing processes $(K_t)_{t\leq T}$ such that $K_0=0$ and $K_T<+\infty$, $\mathbb{P}-a.s.$

$\mathcal{K}^2$:= the space of $\mathcal{P}$-measurable continuous nondecreasing processes $(K_t)_{t\leq T}$ such that $K_0=0$ and $\mathbb{E}\left(K_T^2\right)<+\infty$.

For $(a,b)\in\mathbb{R}^2$, we denote $a\vee b := \max(a,b)$, $a\wedge b := \min(a,b)$, $a^- := \max(0,-a)$ and $a^+ := \max(0,a)$.

We consider the following assumptions on the data.

(A.1) $H(t,\omega,y,z)$ is a real valued, $\mathcal{F}_t$-progressively measurable process defined on $[0,T]\times\Omega\times\mathbb{R}\times\mathbb{R}^d$.

(A.2) ξ is a square integrable, $\mathcal{F}_T$-measurable random variable defined on $(\Omega,\mathcal{F},\P)$.

(A.3) $U=(U_t)_{0\leq t\leq T}$ and $L=(L_t)_{0\leq t\leq T}$ are two processes which belong to $\mathcal{S}^2$ such that $\forall t\leq T$, $L_t<U_t$ and $L_T\leq\xi\leq U_T$, $\mathbb{P}-a.s.$

Definition 1.1. A solution of a BSDE with two reflecting barriers, with the data (ξ,H,L,U), is an $(\mathcal{F}_t)$-adapted process $(Y,Z,K^+,K^-) := \left(Y_t,Z_t,K_t^+,K_t^-\right)_{0\leq t\leq T}$ which satisfies the following equation.

$$\begin{cases} (\mathbf{i})\ Y \in \mathcal{C}, K^+ \text{ and } K^- \in \mathcal{K},\ Z \in \mathcal{L}^2, \\ (\mathbf{ii})\ Y_t = \xi + \int_t^T H(s, Y_s, Z_s)ds + \int_t^T dK_s^+ - \int_t^T dK_s^- - \int_t^T Z_s dB_s, \quad 0 \le t \le T, \\ (\mathbf{iii})\ \forall t \le T,\ L_t \le Y_t \le U_t, \\ (\mathbf{iv})\ \int_0^T (U_t - Y_t)dK_t^- = \int_0^T (Y_t - L_t)dK_t^+ = 0. \end{cases} \tag{1.1}$$

ξ is called the terminal value, H the generator or coefficient, L the lower barrier and U the upper barrier.

In the sequel the previous equation will be labeled $eq(H, \xi, L, U)$ or $bsde(H, \xi, L, U)$.

The backward stochastic differential equations with two reflecting barriers (RBSDEs) have been introduced by Civitanic and Karatzas [7]. A RBSDE (1.1) is called quadratic if its generator has at most a quadratic growth in its z variable. In [7], the authors have proved the existence and uniqueness of the solution in the case of a uniformly Lipschitz generator H and a square terminal condition ξ. Moreover, either the barriers are regular or they satisfy the so-called Mokobodski's condition which leads to the existence of a difference of non-negative super-martingales between L and U. When ξ is bounded, H satisfies $H(s, y, z) \le C(1+\phi(y)+|z|^2)$ [with ϕ is some function which is bounded on compact sets] and the barriers satisfy the Mokobodski's condition, the existence of a solution have been proved, it has been shown in Bahlali, Hamadène and Mezerdi [4]. In Hamadène and Hassani [9], the existence of solutions to RBSDE (1.1) has been proved in the case where the terminal value ξ is square integrable and H has a uniform linear growth in y and z and the barriers are square integrable and satisfying $L_t < U_t$, $\forall t \in [0, T]$. In Lepeltier and San Martin [12], the existence of solutions to RBSDE (1.1) has been established under a linearly increasing generator H and a square integrable terminal data ξ. Essaky and Hassani [8] considered RBSDE (1.1) in a very general situation. Indeed, they show the existence of a minimal and a maximal solution when ξ is only $\mathcal{F}_T-$measurable and H has a general growth with respect to the variable y and stochastic quadratic growth with respect to the variable z. More precisely, Essaky and Hassani assume that there exist a $\mathcal{F}_t$-adapted processes η such that $\mathbb{E}\left(\int_0^T \eta_s ds\right) < +\infty$ and a continuous process C such that $|H(t, y, z)| \le \eta_t(\omega) + C_t(\omega)\,|z|^2$, and there exists a continuous semimartingale whic passes between L and U. The latter is satisfied, for instance, when $L < U$.

The main objective of this work is to extend the result of [1, 2] to quadratic reflected BSDEs. We first use the occupation time formula to show that for any solution (Y, Z, K^+, K^-) of RBSDE (1.1) with data $(H(t, y, z), \xi, L, U)$ such that there exist a locally integrable positive function f and $\mathcal{F}_t$-adapted positive stochastic process η satisfying $\mathbb{E}\int_0^T \eta_s ds < +\infty$ such that for $a.e.$ (t, ω) and every (y, z),

$$|H(t, y, z)| \le \eta_t + f(y)|z|^2,$$

then the time spent by Y in a Lebesgue negligible set is negligible with respect to the measure $|Z_t|^2 dt$. i.e. the following Krylov's estimate holds for any nonnegative measurable function Ψ

$$\mathbb{E}\int_0^{T\wedge\tau_R} \Psi(Y_s)\,|Z_s|^2\,ds \le C\,\|\Psi\|_{\mathbb{L}^1([R,-R])}, \tag{1.2}$$

where τ_R is the first exit time of Y from the interval $[-R, R]$ and C is a constant depending on T, $\|\eta\|_{\mathbb{L}^1(\Omega)}\ \mathbb{E}\left[K_T^{\pm}\right]$ and $\|f\|_{\mathbb{L}^1([R,-R])}$.

Using inequality (1.2) we prove that the following Itô–Krylov's change of variable formula holds for Φ belonging to the Sobolev space $\mathcal{W}^2_{p,loc}$

$$\Phi(Y_t) = \Phi(Y_0) + \int_0^t \Phi'(Y_s)dY_s + \frac{1}{2}\int_0^t \Phi''(Y_s)\,|Z_s|^2\,ds. \tag{1.3}$$

where (Y, Z, K^+, K^-) is an arbitrary solution to the RBSDE (1.1). This allows us to establish the existence of solution to the RBSDE (1.1).

The Paper is organized as follows. In section 2 (auxilliary results), we present the results of Hamadène & Hassani [9] which will be used. In section 3, we consider the reflected BSDEs

$bsde(f(y)\,|z|^2, \xi, L, U)$ and $bsde(\phi_f(y,z), \xi, L, U)$, where a, b, c are real numbers and

$$\phi_f(y,z) := a + b\,|y| + c\,|z| + f(y)\,|z|^2 . \tag{1.4}$$

Assuming that f is continuous and globally integrable on $\mathbb{R}$, ξ is square integrable, we establish the existence of solutions for the two equations. The uniqueness is also established for $bsde(f(y)\,|z|^2, \xi, L, U)$.

In section 4, we prove Krylov's inequality and Itô–Krylov's change of variable formula for the solutions of BSDEs with two reflecting barriers then we use them to extend the results of sections 3 to the case where f is merely integrable on $\mathbb{R}$.

2 Auxiliary results

Consider the following assumptions.

(A.4) for any t, y, y', z, z', there exists a constant $C \geq 0$ such that $\mathbb{P}-a.s.$

$$|H(s,y,z) - H(s,y',z')| \leq C\left(|y-y'| + |z-z'|\right).$$

(A.5) H is continuous and has a uniform linear growth in y and z, i.e. there exists a constant C such that

$$|H(s,y,z)| \leq C\left(1 + |y| + |z|\right).$$

The following results is established in [9]. The authors proved the existence and uniqueness of the solution to reflected BSDEs (1.1) when $L_t < U_t$, for each $t \leq T$.

Theorem 2.1. *([9], Theorem 3.7) Let assumptions* $(\mathbf{A.1}) - (\mathbf{A.4})$ *be hold. Then, there exists a unique solution* (Y,Z,K) *to the equation (1.1) which belongs to* $\mathcal{S}^2 \times \mathcal{M}^2 \times \mathcal{K}^2$.

Theorem 2.2. *([9], Theorem 5.1) Assume that* $(\mathbf{A.1}) - (\mathbf{A.3})$ *and* $(\mathbf{A.5})$ *are satisfied. Then, equation (1.1) has at least one solution* (Y,Z,K) *which belongs to* $\mathcal{S}^2 \times \mathcal{M}^2 \times \mathcal{K}^2$.

3 Reflected BSDE with continuous generators

In this section, we will study the two types of the reflected BSDE (1.1) with data $(f(y)\,|z|^2, \xi, L, U)$ and $(\phi_f(y,z), \xi, L, U)$. Assuming that f is continuous and globally integrable on $\mathbb{R}$ and ξ square integrable. The following lemma is necessary to study the two types of the reflected quadratic BSDEs. It allows us to eliminate the additive quadratic term.

Lemma 3.1. *([2], Lemma **A.1**). Let f be continuous and belongs to $\mathbb{L}^1(\mathbb{R})$. The function*

$$u(x) := \int_0^x \exp\left(2\int_0^y f(t)\,dt\right) dy,$$

has the following properties,

$(\mathbf{i})$ *$u \in C^2(\mathbb{R})$ and satisfies the differential equation $\frac{1}{2}u''(x) - f(x)u'(x) = 0$ on $\mathbb{R}$.*

$(\mathbf{ii})$ *u is a one to one function from $\mathbb{R}$ onto $\mathbb{R}$.*

$(\mathbf{iii})$ *The inverse function u^{-1} belongs to $C^2(\mathbb{R})$.*

$(\mathbf{iv})$ *u is a quasi-isometry, that is there exist two positive constants m and M such that, for any $x, y \in \mathbb{R}$,*

$$m\,|x-y| \leq |u(x) - u(y)| \leq M\,|x-y|. \tag{3.1}$$

We explain how we establish the existence of a solution for both $bsde(f(y)\,|z|^2, \xi, L, U)$ and $bsde(\phi_f(y,z), \xi, L, U)$. Let u be the function defined in Lemma 3.1, we define the processes $\overline{\xi}$, $\overline{L}$, $\overline{S}$, $\overline{Y}$, $\overline{Z}$, $\overline{K}^{\pm}$ and $\overline{H}$ as follows:

$$\begin{cases} \overline{\xi} = u(\xi), \qquad \overline{L}_s = u(L_s), \qquad \overline{S} = u(S_s), \\ \overline{Y}_. = u(Y_.), \quad \overline{Z}_. = u'(Y_.)Z_., \quad d\overline{K}^{\pm}_. = u'(Y_.)dK^{\pm}_., \\ \overline{H}(s,\overline{y},\overline{z}) = u'\left(u^{-1}(\overline{y})\right) H\left(s, u^{-1}(\overline{y}), \left[u^{-1}(\overline{y})\right]'\overline{z}\right) - f\left(u^{-1}(\overline{y})\right)\left[u^{-1}(\overline{y})\right]'|\overline{z}|^2 . \end{cases} \tag{3.2}$$

Assume that (Y, Z, K^+, K^-) is a solution (resp. maximal solution) of the RBSDE (1.1). Then, the following RBSDE has a solution (resp. maximal solution)

$$\begin{cases} \textbf{(i)}\ \overline{Y} \in \mathcal{C}, \overline{K}^+ \text{ and } \overline{K}^- \in \mathcal{K},\ \overline{Z} \in \mathcal{L}^2, \\ \textbf{(ii)}\ \overline{Y}_t = \overline{\xi} + \int_t^T \overline{H}(s, \overline{Y}_s, \overline{Z}_s)ds + \int_t^T d\overline{K}_s^+ - \int_t^T d\overline{K}_s^- - \int_t^T \overline{Z}_s dB_s, \quad 0 \le t \le T, \\ \textbf{(iii)}\ \forall t \le T,\ \overline{L}_t \le \overline{Y}_t \le \overline{U}_t, \\ \textbf{(iv)}\ \int_0^T (\overline{U}_t - \overline{Y}_t) d\overline{K}_t^- = \int_0^T (\overline{Y}_t - \overline{L}_t) d\overline{K}_t^+ = 0. \end{cases} \tag{3.3}$$

Remark 3.2. The maximality property of the solutions is preserved by the fact that the function u and its inverse are strictly increasing.

Proposition 3.3. *Assume that f is continuous and globally integrable on $\mathbb{R}$ and ξ is square integrable on Ω. Then, equation (1.1) has a solution (resp. maximal solution) if and only if equation (3.3) has a solution (resp. maximal solution). Moreover, the solutions belong to $\mathcal{S}^2 \times \mathcal{M}^2 \times \mathcal{K}^2$.*

Proof. Suppose that (Y, Z, K^+, K^-) is a solution (resp. maximal solution) of RBSDE (1.1). Let $u(x) := \int_0^x \exp\left(2\int_0^y f(t)\, dt\right) dy$ be the function defined in Lemma 3.1. We use Itô's formula to show that

$$\begin{aligned} u(Y_t) = u(\xi) &+ \int_t^T u'(Y_s) \left\{ H(s, Y_s, Z_s) - f(Y_s) |Z_s|^2 \right\} ds \\ &+ \int_t^T u'(Y_s)\, dK_s^+ - \int_t^T u'(Y_s)\, dK_s^- - \int_t^T u'(Y_s)\, Z_s dB_s \end{aligned}$$

Since both u and its inverse are $\mathcal{C}^2$ class functions which are globally Lipschitz and one to one from $\mathbb{R}$ onto $\mathbb{R}$, then every solution $\left(\overline{Y}, \overline{Z}, \overline{K}^+, \overline{K}^-\right)$ of equation (3.2) is a solution (resp. maximal solution) of equation (3.3) with data $(\overline{H}(s, \overline{y}, \overline{z}), \overline{\xi}, \overline{L}, \overline{U})$. Conversely, suppose that there exists a solution (resp. maximal solution) $\left(\overline{Y}, \overline{Z}, \overline{K}^+, \overline{K}^-\right)$ for equation (3.3). Hence, according to Lemma 3.1, one can see that (Y_t, Z_t, K_t^+, K_t^-) is a solution (resp. maximal solution) for equation (1.1), where

$$\begin{cases} Y_t = u^{-1}\left(\overline{Y}_t\right), \qquad Z_{\cdot} = \left[u^{-1}\left(\overline{Y}_t\right)\right]' \overline{Z}_t, \\ dK_t^{\pm} = \left[u^{-1}\left(\overline{Y}_t\right)\right]' d\overline{K}_t^{\pm}, \ \text{ for all } t \le T. \end{cases}$$

This shows that, equations (1.1) and (3.3) are equivalent. □

3.1 Reflected Quadratic BSDEs with $H(t, y, z) := f(y)\,|z|^2$

Proposition 3.4. *Let f be a continuous and integrable function. Assume that conditions* $(\mathbf{A.2})$ *and* $(\mathbf{A.3})$ *are satisfied. Then, equation $bsde(f(y)\,|z|^2, \xi, L, U)$ has a unique solution.*

Proof. We use the same notations as in Proposition 3.3. According to Proposition 3.3, the Reflected Quadratic BSDE (RQBSDE) with data $(f(y)\,|z|^2, \xi, L, U)$ has a unique solution if and only if the following equation has a unique solution

$$\begin{cases} \textbf{(i)}\ \overline{Y} \in \mathcal{S}^2, \overline{K}^+ \text{ and } \overline{K}^- \in \mathcal{K}^2,\ \overline{Z} \in \mathcal{M}^2, \\ \textbf{(ii)}\ \overline{Y}_t = \overline{\xi} + \int_t^T d\overline{K}_s^+ - \int_t^T d\overline{K}_s^- - \int_t^T \overline{Z}_s dB_s, \quad 0 \le t \le T, \\ \textbf{(iii)}\ \forall t \le T,\ \overline{L}_t \le \overline{Y}_t \le \overline{U}_t, \\ \textbf{(iv)}\ \int_0^T (\overline{U}_t - \overline{Y}_t) d\overline{K}_t^- = \int_0^T (\overline{Y}_t - \overline{L}_t) d\overline{K}_t^+ = 0. \end{cases} \tag{3.4}$$

By Assumption $(\mathbf{A.2})$ and Lemma 3.1, we have $\overline{\xi} = u(\xi)$ is square integrable. From the result of Theorem 2.1, equation (3.4) has a unique solution. □

Reflected Quadratic BSDE with $H(t,y,z) = a + b|y| + c|z| + f(y)|z|^2 := \phi_f(y,z)$

Let $\phi_f(y,z) := a + b|y| + c|z| + f(y)|z|^2$, a, b, $c \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$. The objective is to prove that the following reflected quadratic BSDE has at least one solution.

$$\begin{cases} \textbf{(i)}\ Y \in \mathcal{S}^2, K^+ \text{ and } K^- \in \mathcal{K}^2,\ Z \in \mathcal{M}^2, \\ \textbf{(ii)}\ Y_t = \xi + \int_t^T \phi_f(y,z)ds + \int_t^T dK_s^+ - \int_t^T dK_s^- - \int_t^T Z_s dB_s, \quad 0 \le t \le T, \\ \textbf{(iii)}\ \forall t \le T,\ L_t \le Y_t \le U_t, \\ \textbf{(iv)}\ \int_0^T (U_t - Y_t)dK_t^- = \int_0^T (Y_t - L_t)dK_t^+ = 0. \end{cases} \tag{3.5}$$

Proposition 3.5. *Let* $(\mathbf{A.2})$, $(\mathbf{A.3})$ *be satisfied. Assume moreover that f is continuous and belongs to* $\mathbb{L}^1(\mathbb{R})$. *Then the reflected quadratic BSDE (3.5) has at least one solution.*

Proof. Let u be the function defined in Lemma 3.1. Let $\overline{Y}$, $\overline{K}^{\mp}$, $\overline{Z}$, $\overline{\xi}$, $\overline{L}$ and $\overline{U}$ defined by (3.2). Consider the BSDE :

$$\begin{cases} \textbf{(i)}\ \overline{Y} \in \mathcal{S}^2, \overline{K}^+ \text{ and } \overline{K}^- \in \mathcal{K}^2,\ \overline{Z} \in \mathcal{M}^2 \\ \textbf{(ii)}\ \overline{Y}_t = \overline{\xi} + \int_t^T G(\overline{Y}_s, \overline{Z}_s)ds + \int_t^T d\overline{K}_s^+ - \int_t^T d\overline{K}_s^- - \int_t^T \overline{Z}_s dB_s, \quad 0 \le t \le T \\ \textbf{(iii)}\ \forall t \le T,\ \overline{L}_t \le \overline{Y}_t \le \overline{U}_t, \\ \textbf{(iv)}\ \int_0^T (\overline{U}_t - \overline{Y}_t)d\overline{K}_t^- = \int_0^T (\overline{Y}_t - \overline{L}_t)d\overline{K}_t^+ = 0, \end{cases} \tag{3.6}$$

where $G(\overline{y},\overline{z}) = (a + b\left|u^{-1}(\overline{y})\right|)u'(u^{-1}(\overline{y})) + c|\overline{z}|$. Using Lemma 3.1, we show that $\overline{\xi}$ is square integrable and G is continuous and of linear growth. Hence, according to Theorem 2.2, the BSDE (3.6) has at least one solution. We use Proposition 3.3 to get the desired result. □

4 Reflected Quadratic BSDEss with measurable coefficient

4.1 Krylov's estimates and Itô–Krylov's formula in equation

We consider the following assumptions:

$(\mathbf{H1})$ There exist a locally integrable nonnegative function f and an $\mathcal{F}_t$-adapted nonnegative stochastic process η satisfying $\mathbb{E}\int_0^T \eta_s ds < +\infty$ such that for $a.e.$ (t,ω) and every (y,z),

$$|H(t,y,z)| \le \eta_t + f(y)|z|^2.$$

$(\mathbf{H2})$ $f \in \mathbb{L}^1(\mathbb{R})$.

Proposition 4.1. *(Local estimte) Let (Y, Z, K^+, K^-) be a solution of the equation (1.1) with data $(H(t,y,z), \xi, L, U)$ and assume that the generator H satisfy Assumption* $(\mathbf{H1})$. *Then, there exists a positive constant C depending on T, R, $\mathbb{E}\left[K_T^{\pm}\right]$ and $\|f\|_{\mathbb{L}^1([R,-R])}$ such that for any positive measurable function Ψ,*

$$\mathbb{E}\int_0^{T\wedge\tau_R} \Psi(Y_s)\,|Z_s|^2\,ds \le C\,\|\Psi\|_{\mathbb{L}^1([R,-R])},$$

where $\tau_R := \inf\{t > 0 : |Y_t| \ge R\}$

Proof. Let $\tau := \tau_R \wedge \tau'_N \wedge \tau''_M \wedge \tau_l^+ \wedge \tau_{l'}^-$, where

$$\tau'_N := \inf\{t > 0 : \int_0^t |Z_s|^2\,ds \ge N\},$$

$$\tau''_M := \inf\{t > 0 : \int_0^t |H(s, Y_s, Z_s)|\,ds \ge M\},$$

$$\tau_l^+ := \inf\{t > 0 : K_t^+ \ge l\},$$

$$\tau_{l'}^- := \inf\{t > 0 : K_t^- \ge l'\}.$$

Clearly, τ'_N tends to infinity as N tends to infinity. The same do for τ_R, τ''_M, τ_l^+ and $\tau_{l'}^-$. For $a \in \mathbb{R}$ such that $a \leq R$, let $\mathbf{L}^a_\cdot(Y)$ be the local time of Y at the level a. Using the occupation time formula, we show that for any nonnegative function Ψ, we have

$$\begin{aligned}\mathbb{E}\left[\int_0^{t\wedge\tau_R} \Psi(Y_s)\,|Z_s|^2\,ds\right] &= \mathbb{E}\left[\int_0^{t\wedge\tau_R} \Psi(Y_s)d\,\langle Y\rangle_s\right] \\ &\leq \mathbb{E}\left[\int_{-R}^{R} \Psi(a)\mathbf{L}^a_{t\wedge\tau_R}(Y)da\right].\end{aligned} \tag{4.1}$$

On the other hand, using Tanaka's formula and the fact that the map $y \to (y-a)^+$ is Lipschitz, we get

$$(Y_{t\wedge\tau} - a)^+ = (Y_{t\wedge\tau} - a)^+ + \int_0^{t\wedge\tau} 1_{\{Y_s \geq a\}} dY_s + \frac{1}{2}\mathbf{L}^a_{t\wedge\tau}(Y).$$

It follows that

$$\begin{aligned}\frac{1}{2}\mathbf{L}^a_{t\wedge\tau}(Y) + \int_0^{t\wedge\tau} 1_{\{Y_s\geq a\}} dK_s^- \leq |Y_{t\wedge\tau} - Y_0| &+ \int_0^{t\wedge\tau} 1_{\{Y_s\geq a\}} H(s,Y_s,Z_s)ds \\ &+ \int_0^{t\wedge\tau} 1_{\{Y_s\geq a\}} dK_s^+ - \int_0^{t\wedge\tau} 1_{\{Y_s\geq a\}} Z_s dW_s.\end{aligned} \tag{4.2}$$

Passing to expectation in the previous inequality, we obtain

$$\sup_a \mathbb{E}\left[\mathbf{L}^a_{t\wedge\tau}(Y)\right] \leq 2\left(M + l + l' + 2R\right). \tag{4.3}$$

Note that $Support(\mathbf{L}^a_{\cdot\wedge\tau}(Y)) \subset [-R, R]$ and $\mathbb{E}\left(K_T^{\pm}\right)^2 < +\infty$. Therefore, using the occupation time formula, we get

$$\begin{aligned}\frac{1}{2}\mathbf{L}^a_{t\wedge\tau}(Y) &\leq |Y_{t\wedge\tau} - Y_0| + \int_0^T \eta_s ds + \int_0^{t\wedge\tau} 1_{\{Y_s\geq a\}} f(Y_s) d\,\langle Y\rangle_s \\ &+ K_T^+ + K_T^- - \int_0^{t\wedge\tau} 1_{\{Y_s\geq a\}} Z_s dW_s \\ &= |Y_{t\wedge\tau} - Y_0| + \int_0^T \eta_s ds + \int_{-R}^{a} f(x)\mathbf{L}^x_{t\wedge\tau}(Y)dx \\ &+ K_T^+ + K_T^- - \int_0^{t\wedge\tau} 1_{\{Y_s\geq a\}} Z_s dW_s.\end{aligned}$$

It follows that

$$\begin{aligned}\mathbb{E}\left[\mathbf{L}^a_{t\wedge\tau}(Y)\right] \leq 2\mathbb{E}&\left[|Y_{t\wedge\tau} - Y_0| + K_T^+ + K_T^- + \int_0^T \eta_s ds\right] \\ &+ \int_{-R}^{a} 2\,|f(x)|\,\mathbb{E}\left[\mathbf{L}^x_{T\wedge\tau}(Y)\right]dx.\end{aligned} \tag{4.4}$$

using Gronwall's lemma, we get

$$\mathbb{E}\left[\mathbf{L}^a_{t\wedge\tau}(Y)\right] \leq C' \exp\left(2\,\|f\|_{\mathbb{L}^1_{([-R.R])}}\right),$$

where $C' = 4R + 2\left[\|\eta\|_{\mathbb{L}^1(\Omega)} + \mathbb{E}\left(K_T^+ + K_T^-\right)\right]$. Passing successively to the limit on N, M, l and l' then using From Beppo–Levi's theorem, we obtain

$$\mathbb{E}\left[\mathbf{L}^a_{t\wedge\tau_R}(Y)\right] \leq C' \exp\left(2\,\|f\|_{\mathbb{L}^1_{([-R.R])}}\right). \tag{4.5}$$

At last, from (4.5) and (4.1), we obtain

$$\mathbb{E}\left[\int_0^{t\wedge\tau_R} \Psi(Y_s)\,|Z_s|^2\,ds\right] \leq C' \exp\left(2\,\|f\|_{\mathbb{L}^1_{([-R.R])}}\right) \|\Psi\|_{\mathbb{L}^1_{([-R.R])}}.$$

Proposition 4.1 is proved. □

Corollary 4.2. *(Global estimate). Assume that* $(\mathbf{H}.1)$ *and* $(\mathbf{H}.2)$ *hold and* ξ *is square integrable. Let* (Y, Z, K^+, K^-) *be a solution of equation (1.1). Then there exists a positive constant* C *depending on* T*,* $\|\eta\|_{\mathbb{L}^1([0,T]\times\Omega)}$ *,* $\mathbb{E}\left[K_T^{\pm}\right]$ *,* $\|f\|_{\mathbb{L}^1(\mathbb{R})}$ *and* $\mathbb{E}\left(\sup_{0\leq t\leq T}|Y_t|\right)$ *such that, for any non-negative measurable function* Ψ*,*

$$\mathbb{E}\left[\int_0^T \Psi(Y_s)\,|Z_s|^2\,ds\right] \leq C\,\|\Psi\|_{\mathbb{L}^1(\mathbb{R})}$$

In particular,

$$\mathbb{E}\int_0^{T\wedge\tau_R} \Psi(Y_s)\,|Z_s|^2\,ds \leq C\,\|\Psi\|_{\mathbb{L}^1([R,-R])}\,,$$

where $\tau_R := \inf\{t > 0 : |Y_t| \geq R\}$.

4.2 Itô-Krylov's formula for Reflected BSDEs.

In this subsection, we establish Itô–Krylov's change of variable formula for the solutions of one dimensional BSDEs with two reflecting barriers. This extends the results obtained in [1, 2].

Theorem 4.3. *Let* (Y, Z, K^+, K^-) *be a solution of the BSDE with two reflecting barriers (1.1). Assume that* $(\mathbf{H}.1)$ *is satisfied and* ξ *is square integrable. Then, for any function* $u \in \mathcal{W}^2_{1,loc}$*, we get*

$$u(Y_t) = u(Y_0) + \int_0^t u'(Y_s)dY_s + \frac{1}{2}\int_0^t u''(Y_s)\,|Z_s|^2\,ds. \tag{4.6}$$

Remark 4.4. Note that, any function of $\mathcal{W}^2_{1,loc}(\mathbb{R})$ have a representant which belongs to $\mathcal{C}^1(\mathbb{R})$. This representant will be considered from now on.

Proof. (of Theorem 4.3) By Proposition 4.1, the term $\int_0^t u''(Y_s)\,|Z_s|^2\,ds$ is well defined. Using classical regularization by convolution, we consider the sequence $(u_n)_n$, such that for any u_n of $\mathcal{C}^2$-class functions satisfying :

(i) u_n converges uniformly to u in the interval $[-R, R]$.

(ii) u'_n converges uniformly to u' in the interval $[-R, R]$.

(iii) u''_n converges in $\mathbb{L}^1([-R, R])$ to u''.

For $R > |Y_0|$, let $\tau_R := \inf\{t > 0 : |Y_t| \geq R\}$. Since τ_R tends to infinity as R tends to infinity, it is enough to establish formula (4.6) for $Y_{t\wedge\tau_R}$. Itô's formula applied to $u_n(Y_{t\wedge\tau_R})$ shows that

$$u_n(Y_{t\wedge\tau_R}) = u_n(Y_0) + \int_0^{t\wedge\tau_R} u'_n(Y_s)dYs + \frac{1}{2}\int_0^{t\wedge\tau_R} u''_n(Y_s)\,|Z_s|^2\,ds \tag{4.7}$$

Passing to the limit on n in the equation (4.7) then using properties (**i**), (**ii**), (**iii**) and Proposition 4.1, we get

$$u(Y_{t\wedge\tau_R}) = u(Y_0) + \int_0^{t\wedge\tau_R} u'(Y_s)dY_s + \frac{1}{2}\int_0^{t\wedge\tau_R} u''(Y_s)\,|Z_s|^2\,ds.$$

□

4.3 Reflected quadratic BSDE with measurable generators

We will use the Itô–Krylov formula (4.6) to study the existence of solutions for the two classes of Reflected Quadratic BSDEs (1.1) with data $(f(y)\,|z|^2, \xi, L, U)$ and $(\phi_f(y, z), \xi, L, U)$. We also prove a comparison theorem for equation (1.1) with data $(f(y)\,|z|^2, \xi, L, U)$.

Lemma 4.5. *([2], Lemma **A.1**). Let f belongs to $\mathbb{L}^1(\mathbb{R})$. The function*

$$u(x) := \int_0^x \exp\left(2\int_0^y f(t)\,dt\right) dy$$

has the following properties,

(**i**) *$u \in \mathcal{W}^2_{1,loc}(\mathbb{R})$ and satisfies the differential equation $\frac{1}{2}u''(x) - f(x)u'(x) = 0$ a.e on $\mathbb{R}$.*

(**ii**) *u is a one to one function from $\mathbb{R}$ onto $\mathbb{R}$.*

(**iii**) *The inverse function u^{-1} belongs to $\mathcal{W}^2_{1,loc}(\mathbb{R})$.*

(**iv**) *u is a quasi-isometry, that is there exist two positive constants m and M such that, for any $x, y \in \mathbb{R}$,*

$$m\,|x-y| \le |u(x) - u(y)| \le M\,|x-y|\,. \tag{4.8}$$

Remark 4.6. Since f is not continuous, then function $u(x) := \int_0^x \exp\left(2\int_0^y f(t)\,dt\right) dy$ is not of class $\mathcal{C}^2$. Therefore, the classical Itô's formula can not be applied. But since u belongs to the Sobolev space $\mathcal{W}^2_{1,loc}$, we can use the Itô-Krylov formula.

The Reflected Quadratic BSDE with $H(t,y,z) := f(y)\,|z|^2$

The following result gives the existence and uniqueness of the solution to $bsde(f(y)\,|z|^2, \xi, L, U)$.

Proposition 4.7. *Assume that* $(\mathbf{A.2})$ *and* $(\mathbf{H.2})$ *are satisfied. Then,* $bsde(f(y)\,|z|^2, \xi, L, U)$ *has a unique solution.*

Proof. Let u be the function defined in Lemma 4.5. Thanks to Theorem 2.1 the RBSDEs (3.3) with the data $(0, \overline{\xi}, \overline{L}, \overline{U})$ has a unique solution. Since u^{-1} belongs to $\mathcal{W}^2_{1,loc}(\mathbb{R})$, then Itô–Krylov formula (3.3), applied to u^{-1} and the solution of equation (3.3) with data $(0, \overline{\xi}, \overline{L}, \overline{U})$, allows us to get the desired result. □

The following proposition shows that we can compare the solutions for the two reflecting barriers quadratique BSDEs with the generators $H(t,y,z) := f(y)\,|z|^2$ although the generators are not continuous the comparison holds.

Proposition 4.8. *(Comparison) Let f_1, f_2 be globally integrable on $\mathbb{R}$. Let ξ_1, ξ_2 belongs to $\mathbb{L}^2$. Let $(Y^{f_1}, Z^{f_1}, K^{+,f_1}, K^{-,f_1})$ and $(Y^{f_2}, Z^{f_2}, K^{+,f_2}, K^{-,f_2})$ be solutions of RQBSDE (**??**) with data $(f_1(y)\,|z|^2, \xi_1, L^{f_1}, U^{f_1})$ and $(f_2(y)\,|z|^2, \xi_2, L^{f_2}, U^{f_2})$, respectively. Assume that $\xi_1 \le \xi_2$ a.s, $f_1 \le f_2$ a.e. and for any $t \le T$, $L^{f_1}_t \le L^{f_2}_t$, $U^{f_1}_t \le U^{f_2}_t$ a.s. Then $Y^{f_2}_t \ge Y^{f_1}_t$ for all t $\mathbb{P}-a.s.$*

Proof. For any function f belongs to $\mathbb{L}^1(\mathbb{R})$, Consider $u_f(x) := \int_0^x exp\left(2\int_0^y f(t)dt\right) dy$. Using Theorem (4.3) to the function $u_{f_1}(Y^{f_2}_t)$, we have

$$u_{f_1}(Y^{f_2}_T) + \int_t^T u'_{f_1}(Y^{f_2}_s)dK^{+,f_2}_s = u_{f_1}(Y^{f_2}_t) - \int_t^T u'_{f_1}(Y^{f_2}_s)f_2(Y^{f_2}_s)\left|Z^{f_2}_s\right|^2 ds + \int_t^T u'_{f_1}(Y^{f_2}_s)dK^{-,f_2}_s$$
$$+ \int_t^T \frac{1}{2}u''_{f_1}(Y^{f_2}_T)\left|Z^{f_2}_s\right|^2 ds + \int_t^T u'_{f_1}(Y^{f_2}_s)Z^{f_2}_s dB_s.$$

Since $u'_{f_1}(x) \ge 0$, we use Lemma 4.5 (**i**) to deduce that

$$u_{f_1}(Y^{f_2}_T) + \int_t^T u'_{f_1}(Y^{f_2}_s)dK^{+,f_2}_s = u_{f_1}(Y^{f_2}_t) - \int_t^T u'_{f_1}(Y^{f_2}_s)\left[f_2(Y^{f_2}_s) - f_1(Y^{f_2}_s)\right]\left|Z^{f_2}_s\right|^2 ds$$
$$+ \int_t^T u'_{f_1}(Y^{f_2}_s)dK^{-,f_2}_s + \int_t^T u'_{f_1}(Y^{f_2}_s)Z^{f_2}_s dB_s.$$

Using to Lemma 4.5 (**iv**) and $f_2(Y^{f_2}_s) - f_1(Y^{f_2}_s) \ge 0$, then there exists two positive constants m and M, such that

$$u_{f_1}(Y^{f_2}_T) - u_{f_1}(Y^{f_2}_t) + m\left(K^{+,f_2}_T - K^{+,f_2}_t\right) \le \int_t^T u'_{f_1}(Y^{f_2}_s)Z^{f_2}_s dB_s + M\left(K^{-,f_2}_T - K^{-,f_2}_t\right).$$

Since u_{f^1} is an increasing function, then using the term $\xi_1 - \xi_2 \leq 0$, we get

$$\begin{aligned} u_{f_1}(Y_t^{f_1}) - u_{f_1}(Y_t^{f_2}) &= \mathbf{E}\left[u_{f_1}(Y_T^{f_1}) \mid \mathcal{F}_t\right] - u_{f_1}(Y_t^{f_2}) \\ &= \mathbf{E}\left[u_{f_1}(\xi_1) \mid \mathcal{F}_t\right] - u_{f_1}(Y_t^{f_2}) \\ &\leq \mathbf{E}\left[u_{f_1}(\xi_2) \mid \mathcal{F}_t\right] - u_{f_1}(Y_t^{f_2}) \\ &\leq \mathbf{E}\left[u_{f_1}(Y_T^{f_2}) - u_{f_1}(Y_t^{f_2}) \mid \mathcal{F}_t\right] \leq 0. \end{aligned}$$

At last, taking $u_{f_1}^{-1}$ to get the desired result. Proposition (4.8) is proved. □

4.4 The reflected BSDE with $H(t, y, z) = a + b\,|y| + c\,|z| + f(y)\,|z|^2 := \phi_f(y, z)$

In this subsection, we will establish the existence of solutions for the Reflected quadratic BSDE (1.1) when $H(t, y, z) := \phi_f(y, z)$, with

$$\phi_f(y, z) := a + b\,|y| + c\,|z| + f(y)\,|z|^2\,,$$

where a, b and c are some nonnegative constants.

Proposition 4.9. *Assume that f is globally integrable on $\mathbb{R}$ and ξ square integrable. Then, the Reflected quadratic BSDE (1.1) with the data $(\phi_f(y, z), \xi, L, U)$ has at least one solution.*

Proof. Let $u(x) := \int_0^x \exp\left(2\int_0^y f(t)\,dt\right) dy$. Using the quasi isometry property of u in Lemma 4.5, we see that ξ is square integrable if and only if $\overline{\xi} := u(\xi)$ is square integrable. Since u is a quasi isometry and u' is bounded, we deduce that $G(\overline{y}, \overline{z}) = (a + b\left|u^{-1}(\overline{y})\right|)u'(u^{-1}(\overline{y})) + c\,|\overline{z}|$ is continuous and with linear growth. Therefore, according to Theorem 2.2, the RBSDE (1.1) with the data $(G(\overline{y}, \overline{z}), \overline{\xi}, \overline{L}, \overline{U})$ has at least one solution. Applying Itô–Krylov's formula (4.6) to the function $u^{-1}(\overline{Y}_t)$, we get that the BSDE (1.1) with the data $(\phi_f(y, z), \xi, L, U)$ has at least one solution. □

Remark 4.10. It is clear that the conclusion of Proposition 4.9 remains true when the generator $\phi_f(y, z)$ is replaced by the generator $\psi_f(y, z) := a + by + cz + f(y)\,|z|^2$.

4.5 The Reflected Quadratic BSDE with the generator H

Proposition 4.11. *Assume that the coefficient $H(t, \omega, y, z)$ is continuous in (y, z) for a.e (t, ω). Assume moreover that $(\mathbf{A.1})$, $(\mathbf{A.2})$, $(\mathbf{A.3})$, $(\mathbf{H.1})$ and $(\mathbf{H.2})$ are satisfied. Then, the reflected quadratic BSDE (1.1) has a minimal and a maximal solution. Moreover, for any solution (Y, Z, K^+, K^-), the component Y belongs to $\mathcal{S}^2$.*

Proof. According to [1], since f is locally bounded, then it can be majorized by a piecewise constant function which we denote h. Therefore, using a suitable interpolation, one can construct a continuous function g such that $g \geq h \geq f$. Note moreover that $(\mathbf{A.3})$ implies that there exists a continuous $(\mathcal{F}_t)$-adapted semimartigale. Therefore, according to Theorem 3.2 of [8] with

$$\eta_t = a + b(|L_t| + |U_t|) + c^2$$

and

$$C_t = 1 + \sup_{s \leq t} \sup_{\alpha \in [0,1]} |g(\alpha L_s + (1 - \alpha)U_s)|,$$

the reflected BSDE (1.1) has a solution (Y, Z, K^+, K^-) such that (Y, Z) belongs to $\mathcal{C} \times \mathcal{L}^2$ and K^+, K^- belong to $\mathcal{K}$. Since L and U belong to $\mathcal{S}^2$, so does for Y because $L \leq Y \leq U$. □

Remark 4.12. We conjecture that under assumptions of Proposition 4.11, one can show that Z belongs to $\mathcal{M}^2$ and $(K^+ - K^-)$ belongs to $\mathcal{K}^2$.

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Author information

Roubi Abdallah, Université de Toulon, IMATH, EA 2134, 83957 La Garde, **France** and Université Med Khider Département de Maths, B.P. 145 Biskra, Algérie.
E-mail: abdallah.roubi@univ-biskra.dz

Labed Boubakeur, Université Med Khider Département de Maths, B.P. 145 Biskra, Algérie.
E-mail: b.labed@univ-biskra.dz

Khaled Bahlali, Université de Toulon, IMATH, EA 2134, 83957 La Garde, France.
E-mail: bahlali@univ-tln.fr

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