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On 2-nil-regular rings

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Abstract We call an element in a ring 2-nil-regular if, it is sum of two regular elements and a nilpotent element. A ring R is called 2-nil-regular if, its all elements are 2-nil-regular. We prove that R is a 2-nil-regular if and only if all Pierce stalks of R are 2-nil-regular. A nonzero ring R is a local ring if and only if R is a 2-nil-regular ring with the only idempotents 0 and 1. Also, for a subring B of a ring A, we prove that $R[A, B]$ is a 2-nil-regular ring if and only if A and B are 2-nil-regular rings.

1 Introduction

In [12], Nicholson et.al. introduced the idea of a clean ring. According to them, an element in a ring R is said to be *clean* if, it is the sum of an idempotent and a unit. A ring R is called *clean* if, all its elements are clean. During the period many authors attracted towards this notion and have worked on it and its generalizations. Recall from [6], an element $r \in R$ is called *nil clean* if there is an idempotent $e \in R$ and a nilpotent $b \in R$ such that $r = e + b$. A ring R is called nil clean if every one of its elements is nil clean. Every nil clean ring is a clean ring by [6, Proposition 3.4]. In [14], an element $a \in R$ is said to be *2-good* if it is sum of two units. A ring R is said to be *2-good* if every element of R is 2-good.

In [1], Abdolyousefi et.al. introduced the idea of a 2-nil-good ring. According to them, a ring R is defined to be *2-nil-good* if every element in R is the sum of two units and a nilpotent.

Recall from [15], a ring R is regular if, for each a in R there exists an $x \in R$ such that $a = axa$. We know that the sum of two regular elements need not be regular. For example, 1 is regular in $\mathbb{Z}_4$ but $1 + 1 = 2$ is not regular in $\mathbb{Z}_4$. By the motivation, we introduce the concept of 2-nil-regular rings. An element $x \in R$ is said to be *2-nil-regular* if, it is sum of two regular elements and a nilpotent element. A ring R is said to be 2-nil-regular if, its all elements are 2-nil-regular.

All regular rings and 2-nil-good rings are 2-nil-regular but not conversely. In support, we give some examples and facts. In the present work, we generalize some well known results and provide various new facts, characterizations and extensions of 2-nil-regular rings.

Throughout, all rings are associative with unity unless otherwise stated. We denote the set of all regular elements, Jacobson radical, prime radical, set of all nilpotent elements, set of all idempotent elements and set of units of a ring R by $reg(R)$, $J(R)$, $P(R)$, $N(R)$, $E(R)$ and $U(R)$, respectively. $T_n(R)$ denotes the ring of all upper triangular matrices over a ring R and C_n is the cyclic group of order n. Let G be a group and R be a ring, we denote the group ring over R by RG. We refer readers to [9] for all undefined terms and notions.

2 Basic properties of 2-nil-regular rings

Remark 2.1. (i) Every regular ring is 2-nil-regular but the converse need not be true. For example, let $R = \mathbb{Z}_4$. Then R is 2-nil-regular but not regular.

(ii) Every clean ring is a 2-nil-regular ring but the converse need not be true. Consider an example from [4, Definition 2.1]. Let F be a field with $char(F) \neq 2$, $A = F[[x]]$ and K be the field of fractions of A. All the ideals of A are generated by power of x, denote by (x^n). Now, if A_F denotes the vector space A over F, define

$R = \{r \in \text{End}(A_F) :$ there exists $q \in K$and a positive integer n, with $r(a) = qa$, for all $a \in (x^n)\}$. Since it is regular, therefore it is 2-nil-regular. But it is not a clean ring.

(iii) A 2-nil-good ring is a 2-nil-regular ring but the converse need not be true. For example, $\mathbb{Z}_6$ is a 2-nil-regular ring but not a 2-nil-good ring as 5 in $\mathbb{Z}_6$ can not be written as sum of two units and a nilpotent. Also, a 2-nil-regular ring need not be 2-good by this example.

Following [7], an associative ring R is said to satisfy unit 1-stable range if $aR+bR = R$ with $a, b \in R$ implies that there exists a $u \in U(R)$ such that $a + bu \in U(R)$.

Proposition 2.2. Every ring satisfying unit 1-stable range is 2-nil-regular.

Proof. Suppose that a ring R satisfies unit 1-stable range and $a \in R$. Let $b = 1$. Then $aR + 1.R = R$. Thus $\exists\ u \in U(R)$ such that $a + 1.u \in U(R)$. We have $a = v + (-u) + 0$, where $v, (-u) \in U(R)$. It follows that a is 2-nil-regular. Hence R is a 2-nil-regular ring. □

Corollary 2.3. Let R be an algebraic algebra over an infinite field. Then R is a 2-nil-regular ring.

Proof. It follows from Proposition 2.2 and [7, Theorem 3.1]. □

Proposition 2.4. A finite product of rings $\prod_{i=1}^{n} R_i$ is a 2-nil-regular ring if and only if each ring R_i is a 2-nil-regular ring. Also, 2-nil-regular rings are closed under homomorphic images.

Recall from [15], let $S(R)$ be the nonempty set of all proper ideals of R generated by central idempotents. The factor ring R/P is called a Pierce stalk of R if P is a maximal element in $S(R)$. We generalize [15, Proposition 2.15] as follows:

Theorem 2.5. The following are equivalent for a ring R:

(i) R is a 2-nil-regular ring.

(ii) All factor rings of R are 2-nil-regular.

(iii) All indecomposable factor rings of R are 2-nil-regular.

(iv) All Pierce stalks of R are 2-nil-regular.

Proof. $(1) \implies (2) \implies (3)$ and $(1) \implies (2) \implies (4)$ are clear.

$(4) \implies (1)$. Suppose that R is not a 2-nil-regular ring. Let C be the set of all proper ideals J generated by central idempotents of R such that R/J is not 2-nil-regular. Then C is nonempty as $(0) \in C$. Since union of each ascending chain of ideals from C is contained in C, therefore C has a maximal element M by Zorn's Lemma. Now, we prove that R/M is a Pierce stalk. In contrary suppose that R/M is not a Pierce stalk. Then there exists a central idempotent f of R such that $M + fR$ and $M + (1 - f)R$ are proper ideals of R which properly contain M. Now, $R/M \simeq R/(M + fR) \times R/(M + (1 - f)R)$. Since $M + fR$ and $M + (1 - f)R$ properly contain M, therefore $M + fR$ and $M + (1-f)R$ are not in C. Thus $R/(M + fR)$ and $R/(M+(1-f)R)$ are 2-nil-regular rings. Hence by Proposition 2.4, R/M is 2-nil-regular which is a contradiction. Then R/M is a Pierce stalk. By hypothesis, R/M is 2-nil-regular which is a contradiction. Hence R is a 2-nil-regular ring.

$(3) \implies (1)$. It is similar to the proof of $(4) \implies (1)$. □

If R is a 2-nil-regular ring and I is an ideal of R, then I need not be nil. For example, $\mathbb{Z}_6$ is a 2-nil-regular ring but the ideal $< 2 >$ is not nil in $\mathbb{Z}_6$. If R/I is a 2-nil-regular ring, then R need not be 2-nil-regular. For example, $\mathbb{Z}/2\mathbb{Z}$ is 2-nil-regular, but $\mathbb{Z}$ is not 2-nil-regular and $2\mathbb{Z}$ is not nil in $\mathbb{Z}$. In the following, we provide a sufficient condition.

Proposition 2.6. Let I be a nil ideal of a ring R. Then R is a 2-nil-regular ring if and only if R/I is a 2-nil-regular ring.

Proof. Let R/I be a 2-nil-regular ring and $a \in R$. Then $a + I = (r_1 + I) + (r_2 + I) + (n + I)$, where $(r_1 + I), (r_2 + I) \in reg(R/I)$ and $(n + I) \in N(R/I)$. Since $r_1 + I \in reg(R/I)$, therefore there exists $y + I \in reg(R/I)$ such that $(r_1 + I)(y + I)(r_1 + I) = r_1 + I$. This implies that

$r_1yr_1 - r_1 \in I$. Also $r_1 - r_1yr_1 \in I$. Since I is a nil ideal of R, therefore idempotents lift modulo I by [6, Proposition 3.15]. Also regular elements lift modulo I by [10, Lemma 2.4]. Then there exists $r \in reg(R)$ such that $r_1 - r \in I$. This implies that $r_1 \in reg(R) + I$. Similarly, $r_2 \in reg(R) + I$. It follows that $a - (r_1 + r_2)$ is nilpotent modulo I and I is nil, then $a - (r_1 + r_2)$ is nilpotent. Hence R is a 2-nil-regular ring. Converse is clear from Proposition 2.4. □

Recall from [1], the prime radical of a ring R is the intersection of all prime ideals of R, denoted by $P(R)$.

Corollary 2.7. The following are equivalent for a ring R:

(i) R is 2-nil-regular.

(ii) $R/P(R)$ is 2-nil-regular.

Proof. Clear by Proposition 2.6. □

Remark 2.8. By [13], an element a in a unital ring R is called *quasi-regular* if, $1 - a$ is being invertible in $(R, \cdot)$. A 2-nil-regular element need not be a quasi-regular element. In $\mathbb{Z}_4$, 3 is not a quasi-regular element but it is a 2-nil-regular element.

3 Extensions of 2-nil-regular rings

Proposition 3.1. Let $P = \begin{pmatrix} R & M \\ 0 & T \end{pmatrix}$ be the formal triangular ring. Then P is a 2-nil-regular ring if and only if R and T are 2-nil-regular rings.

Proof. Let R and T be 2-nil-regular rings and let $\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \in P$. Then

$$\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} = \begin{pmatrix} r_1 + r_1' + n_1 & m \\ 0 & r_2 + r_2' + n_2 \end{pmatrix} = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix} + \begin{pmatrix} r_1' & 0 \\ 0 & r_2' \end{pmatrix} + \begin{pmatrix} n_1 & m \\ 0 & n_2 \end{pmatrix}.$$

Since r_1 and r_2 are regular in R and T respectively, therefore there exists x_1 and x_2 such that $r_1x_1r_1 = r_1$ and $r_2x_2r_2 = r_2$. Then $\begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}\begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix}\begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix} = \begin{pmatrix} r_1x_1r_1 & 0 \\ 0 & r_2x_2r_2 \end{pmatrix} = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}$. Thus $\begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix} \in reg(P)$. Similarly, $\begin{pmatrix} r_1' & 0 \\ 0 & r_2' \end{pmatrix} \in reg(P)$ and $\begin{pmatrix} n_1 & m \\ 0 & n_2 \end{pmatrix} \in N(P)$. Hence P is a 2-nil-regular ring. Conversely, let $\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \in P$. Then $\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} = \begin{pmatrix} r_1 & r_2 \\ 0 & r_3 \end{pmatrix} + \begin{pmatrix} r_1' & r_2' \\ 0 & r_3' \end{pmatrix} + \begin{pmatrix} n_1 & m \\ 0 & n_3 \end{pmatrix}$, where $\begin{pmatrix} r_1 & r_2 \\ 0 & r_3 \end{pmatrix}$ and $\begin{pmatrix} r_1' & r_2' \\ 0 & r_3' \end{pmatrix} \in reg(P)$ and $\begin{pmatrix} n_1 & m \\ 0 & n_3 \end{pmatrix} \in N(P)$. Then $a = r_1 + r_1' + n_1$ and $b = r_3 + r_3' + n_3$. Since $\begin{pmatrix} r_1 & r_2 \\ 0 & r_3 \end{pmatrix} \in reg(P)$, therefore there exists $\begin{pmatrix} y_1 & y_2 \\ 0 & y_3 \end{pmatrix} \in P$ such that $\begin{pmatrix} r_1 & r_2 \\ 0 & r_3 \end{pmatrix}\begin{pmatrix} y_1 & y_2 \\ 0 & y_3 \end{pmatrix}\begin{pmatrix} r_1 & r_2 \\ 0 & r_3 \end{pmatrix} = \begin{pmatrix} r_1 & r_2 \\ 0 & r_3 \end{pmatrix}$. This implies that $r_1y_1r_1 = r_1$ and $r_3y_3r_3 = r_3$. Thus $r_1 \in reg(R)$ and $r_3 \in reg(T)$. Similarly, $r_1' \in reg(R)$ and $r_3' \in reg(T)$. Also $n_1 \in N(R)$ and $n_3 \in N(T)$. Hence R and T are 2-nil-regular rings. □

Corollary 3.2. A ring R is a 2-nil-regular ring if and only if $T_n(R)$ is a 2-nil-regular ring.

Corollary 3.3. Let R be a ring and e be a central idempotent of R. Then R is a 2-nil-regular ring if and only if eRe and $(1 - e)R(1 - e)$ are 2-nil-regular rings.

Proof. Suppose that eRe and $(1-e)R(1-e)$ are 2-nil-regular rings. By [9], the Pierce decomposition for the ring R, $R = eRe \oplus eR(1-e) \oplus (1-e)Re \oplus (1-e)R(1-e)$. Then $R = eRe \oplus (1-e)R(1-e) \simeq \begin{pmatrix} eRe & 0 \\ 0 & (1-e)R(1-e) \end{pmatrix}$ as e is a central idempotent of R. Now, the result follows from Proposition 3.1. Converse is clear from Proposition 2.4. □

Following [11], if R is a ring and $\alpha\colon R \to R$ is a ring endomorphism, then $R[[x,\alpha]]$ denotes the ring of skew formal power series over R, that is, all formal power series in x with cofficients from R with multiplication defined by $xr = \alpha(r)x$ for all $r \in R$. In particular, $R[[x]] = R[[x, 1_R]]$ is the ring of formal power series over R.

Theorem 3.4. The following are equivalent for a ring R:

(i) R is a 2-nil-regular ring.

(ii) $R[x,\alpha]/(x^n)$ is a 2-nil-regular ring.

Proof. $(1) \iff (2)$. Define a map $f\colon P = R[x,\alpha]/(x^n) \to R$ by $f(a_0 + a_1 + \ldots + a_{n-1}x^{n-1} + (x^n)) = a_0$. Clearly, f is a ring epimorphism and $Kerf =< x >$. Then $P/ < x > \simeq R$. Now, it is clear from Proposition 2.6 as $< x >$ is nil in P.

□

Proposition 3.5. Let R be a 2-nil-regular ring. Then

$$S_n(R) = \left\{ \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ 0 & a_0 & a_1 & \cdots & a_{n-2} \\ 0 & 0 & a_0 & \cdots & a_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_0 \end{pmatrix} | a_i \in R\ i = 0, 1, \ldots, n-1 \right\}$$

is a 2-nil-regular ring.

Proof. Since $R[x]/(x^n) \simeq S_n(R)$. Then the result follows from Theorem 3.4. □

Recall from [5] , we say that B is a subring of a ring A if $\phi \neq B \subseteq A$ and for any $x, y \in B$, $x - y, xy \in B$ and $1_A \in B$. Let A be a ring and B be a subring of A and $R[A,B]$ denotes the set

$$\left\{ (a_1, a_2, \ldots, a_n, b, b, \ldots) : a_i \in A, b \in B, 1 \leq i \leq n \right\}.$$

Then $R[A,B]$ is a ring under the componentwise addition and multiplication.

Theorem 3.6. Let A be a ring and B be a subring of A. Then $R[A,B]$ is a 2-nil-regular ring if and only if A and B are 2-nil-regular rings.

Proof. Suppose A and B are 2-nil-regular rings. Let

$$x = (a_1, a_2, \ldots, a_n, b, b, \ldots) \in R[A,B].$$

Now, $a_i = r_i + r_i^{'} + n_i$, where $r_i, r_i^{'} \in reg(A)$ and $n_i \in N(A)$ for each $1 \leq i \leq n$. Also $b = r + r^{'} + n$ where $r, r^{'} \in reg(B)$ and $n \in N(B)$. Then $x = (r_1, r_2, \ldots, r_n, r, r, \ldots)$+$(r_1^{'}, r_2^{'}, \ldots,$ $r_n^{'}, r^{'}, r^{'}, \ldots)$+$(n_1, n_2, \ldots, n_n, n, n, \ldots)$. Since $r_i, r_i^{'} \in reg(A)$, therefore there exists $y_i, y_i^{'} \in A$ such that $r_i y_i r_i = r_i$ and $r_i^{'} y_i^{'} r_i^{'} = r_i^{'}$ and $r, r^{'} \in reg(B)$, then there exists $y, y^{'} \in B$ such that $ryr = r$ and $r^{'} y^{'} r^{'} = r^{'}$. Thus $(r_1, r_2, \ldots, r_n, r, r, \ldots), (r_1^{'}, r_2^{'}, \ldots, r_n^{'}, r^{'}, r^{'}, \ldots) \in reg(R[A,B])$. For each i, $n_i^{m_i} = 0$ and $n^m = 0$ for some $m, m_i \in \mathbb{N}$. Then

$$(n_1, n_2, \ldots, n_n, n, n, \ldots)^{mm_1 m_2 \ldots m_n} = 0.$$

Thus $(n_1, n_2, \ldots, n_n, n, n, \ldots) \in N(R[A,B])$. Hence $R[A,B]$ is a 2-nil-regular ring.

Conversely, suppose that $R[A,B]$ is a 2-nil-regular ring. Let $a \in A$ and write $(a,0,0,\ldots) = (r_1, r_1^{'}, r_1^{'}, \ldots) + (r_2, r_2^{'}, r_2^{'}, \ldots) + (n, n^{'}, n^{'}, \ldots)$, where $(r_1, r_1^{'}, r_1^{'}, \ldots)$, $(r_2, r_2^{'}, r_2^{'}, \ldots) \in reg(R[A,B])$ and $(n, n^{'}, n^{'}, \ldots) \in N(R[A,B])$. We choose elements

$$(y_1, y_1^{'}, y_1^{'}, \ldots), (y_2, y_2^{'}, y_2^{'}, \ldots) \in R[A,B]$$

such that $(r_1, r_1^{'}, r_1^{'}, \ldots)(y_1, y_1^{'}, y_1^{'}, \ldots)(r_1, r_1^{'}, r_1^{'}, \ldots) = (r_1, r_1^{'}, r_1^{'}, \ldots)$ and

$$(r_2, r_2^{'}, r_2^{'}, \ldots)(y_2, y_2^{'}, y_2^{'}, \ldots)(r_2, r_2^{'}, r_2^{'}, \ldots) = (r_2, r_2^{'}, r_2^{'}, \ldots).$$

Then $r_1 y_1 r_1 = r_1$ and $r_2 y_2 r_2 = r_2$, so $r_1, r_2 \in reg(A)$. Also,

$$(n, n^{'}, n^{'}, \ldots)^m = (n^m, n^{'m}, n^{'m}, \ldots) = 0$$

for some $m \in \mathbb{N}$. Then $n \in N(A)$ and $a = r_1 + r_2 + n$. Therefore, A is a 2-nil-regular ring. Now, let $b \in B$ and write

$$(0, b, b, \ldots) = (s_1, s_1^{'}, s_1^{'}, \ldots) + (s_2, s_2^{'}, s_2^{'}, \ldots) + (t, t^{'}, t^{'}, \ldots)$$

where

$$(s_1, s_1^{'}, s_1^{'}, \ldots), (s_2, s_2^{'}, s_2^{'}, \ldots) \in reg(R[A,B])$$

and

$$(t, t^{'}, t^{'}, \ldots) \in N(R[A,B]).$$

Choose $(x_1, x_1^{'}, x_1^{'}, \ldots)$ and $(x_2, x_2^{'}, x_2^{'}, \ldots)$ in $R[A,B]$ such that

$$(s_1, s_1^{'}, s_1^{'}, \ldots)(x_1, x_1^{'}, x_1^{'}, \ldots)(s_1, s_1^{'}, s_1^{'}, \ldots) = (s_1, s_1^{'}, s_1^{'}, \ldots)$$

and

$$(s_2, s_2^{'}, s_2^{'}, \ldots)(x_2, x_2^{'}, x_2^{'}, \ldots)(s_2, s_2^{'}, s_2^{'}, \ldots) = (s_2, s_2^{'}, s_2^{'}, \ldots).$$

Then $s_1^{'}, s_2^{'} \in reg(B)$. Also $t^{'} \in N(B)$ and $b = s_1^{'} + s_2^{'} + t^{'}$. Therefore, B is a 2-nil-regular ring. □

Recall [12], let R be a ring and let M be an (R,R)-bimodule which is a general ring (with or without identity) such that for all $m, l \in M$ and $a \in R$, we have $(ml)a = m(la)$ and $(am)l = a(ml)$. The ideal-extension $I(R,M)$ of R by M is defined as the additive abelian group $I(R,M) = R \oplus M$ with multiplication $(a,m)(c,l) = (ac, al + mc + ml)$.

Proposition 3.7. Let R and M be defined as above. If $mxs + mxm + sxm = m$ for all $m \in M$ and $x, s \in R$, then $I(R,M)$ is a 2-nil-regular ring if and only if R is a 2-nil-regular ring.

Proof. Let $I(R,M)$ be a 2-nil-regular ring. Define $f\colon I(R,M) \to R$ by $f(a,m)$=a. Then clearly, f is a ring epimorphism and by Proposition 2.4, R is a 2-nil-regular ring. Conversely, suppose that R is a 2-nil-regular ring. Let $(a,m) \in I(R,M)$ and $a = r_1 + r_2 + n$ where $r_1, r_2 \in reg(R)$ and $n \in N(R)$. Since $r_1, r_2 \in reg(R)$, therefore there exists $x_1, x_2 \in R$ such that $r_1 x_1 r_1 = r_1$ and $r_2 x_2 r_2 = r_2$. Then $(a,m) = (r_1, m) + (r_2, 0) + (n, 0)$. Now, $(r_1, m)(x_1, 0)(r_1, m) = (r_1 x_1 r_1, r_1 x_1 m + m x_1 r_1 + m x_1 m) = (r_1, m)$. Then $(r_1, m) \in reg(I(R,M))$ by assumption. Also $(r_2, 0) \in reg(I(R,M))$ and $(n, 0) \in N(I(R,M))$. Hence $I(R,M)$ is a 2-nil-regular ring. □

Proposition 3.8. Let R be a ring.

(i) If either $2^{-1} \in R$ or R is a local ring, then RC_2 is a 2-nil-regular ring if and only if R is a 2-nil-regular ring.

(ii) If $3^{-1} \in R$ and $R \subseteq \mathbb{C}$, then RC_3 is a 2-nil-regular ring if and only if R and $R[x]/ < x^2 + x + 1 >$ are 2-nil-regular rings.

Proof. (1). Suppose $2^{-1} \in R$. Then by [17, Lemma 3.1], $RC_2 \simeq R \times R$. Thus RC_2 is a 2-nil-regular ring by Proposition 2.4. Suppose R is a local ring. Then RC_2 is semiperfect and so clean by [3]. Thus RC_2 is a 2-nil-regular ring by Remark 2.1(2). Converse is clear.

(2). Let $3^{-1} \in R$ and $R \subseteq \mathbb{C}$. Then by [17, lemma 3.5], $RC_3 \simeq R \times R[x]/ < x^2 + x + 1 >$. Hence by Lemma 2.4, RC_3 is a 2-nil-regular ring. Converse is clear. □

Proposition 3.9. Let R be a ring with the only idempotents 0 and 1. Then R is a 2-nil-regular ring if and only if R is a 2-nil-good ring.

Proof. Let R be a 2-nil-regular ring. Let $a \in R$ such that $a = r_1 + r_2 + n$, where $0 \neq r_1, 0 \neq r_2 \in reg(R)$ and $n \in N(R)$. Since $r_1 \in reg(R)$, therefore there exists $y \in R$ such that $r_1 y r_1 = r_1$. Then $(r_1 y)^2 = r_1 y r_1 y = r_1 y$, also $(y r_1)^2 = y r_1 y r_1 = y r_1$. So $r_1 y$ and $y r_1$ are idempotents. Since R has only 0 and 1 idempotents, therefore either $r_1 y = y r_1 = 1$ or at least one of $r_1 y$ and $y r_1$ is zero. Suppose at least one of $r_1 y$ and $y r_1$ is zero, then $r_1 = 0$ which is a contradiction. Similarly, we get $r_2 = 0$ which is also a contradiction. Suppose $r_1 y = y r_1 = 1$, then $r_1 \in U(R)$. Similarly, we get $r_2 \in U(R)$. Then from $a = r_1 + r_2 + n$, we have $a \in U(R) + U(R) + N(R)$. Hence R is a 2-nil-good ring. Conversely, let R be a 2-nil-good ring. Let $a \in R$ such that $a = u_1 + u_2 + n$, where $u_1, u_2 \in U(R)$ and $n \in N(R)$. Then $a \in reg(R) + reg(R) + N(R)$ as every unit is regular. Hence R is a 2-nil-regular ring. □

Following [10], a ring R is said to be *Abelian* if, every idempotent in R is central.

Remark 3.10. A 2-nil-regular ring and an Abelian ring do not imply each other. For example,

(i) Let $R = \begin{pmatrix} \mathbb{Z}_4 & \mathbb{Z}_4 \\ 0 & \mathbb{Z}_4 \end{pmatrix}$. Then by Corollary 3.2, R is a 2-nil-regular ring and is not an Abelian ring.

(ii) Let $R = \mathbb{Z}$. Then R is an Abelian ring but not a 2-nil-regular ring.

Lemma 3.11. Let R be an Abelian ring and $r \in reg(R)$. Then $-r$ is clean.

Proof. Let $r \in reg(R)$. Then there exists $x \in R$ such that $rxr = r$. This implies that rx and xr are idempotents. Let $f = rx$. Then $(rf+(1-f))(xf+(1-f)) = rfxf+rf(1-f)+(1-f)xf+1-f = rxf+0+0+1-f$. Since R is Abelian and $rx = f$, therefore $(rf+(1-f))(xf+(1-f)) = 1$. Also, $(xf+(1-f))(rf+(1-f)) = xfrf+xf(1-f)+(1-f)rf+(1-f) = 1$ as R is Abelian. Thus $v = (rf + (1-f))$ is a unit. Now, $fv = frf + f(1-f) = (rx)r(rx) + 0 = rxrxr = r$. Now, let $g = 1-f$. Then $fv+g$ is a unit and also $-(fv+g)$ is a unit. Then $-r = g+(-(fv+g))$. Hence $-r$ is clean as g is an idempotent element. □

Recall from [16], a ring R is called an *exchange ring* if the left regular module R_R has the finite exchange property and showed that this definition is left-right symmetric.

Proposition 3.12. Let R be an Abelian ring and all idempotents of R are orthogonal. Then the following are equivalent:

(i) R is a 2-nil-regular ring.

(ii) R is a nil clean ring.

(iii) R is a clean ring.

(iv) R is an exchange ring.

Proof. $(1) \implies (2)$. Let R be a 2-nil-regular ring. This implies that every $a \in R$ is 2-nil-regular and so $-a$ is 2-nil-regular. Then $-a = r_1 + r_1^{'} + n_1$, where $r_1, r_1^{'} \in reg(R)$ and $n_1 \in N(R)$. This implies that $a = -r_1 + (-r_1^{'}) + (-n_1)$. Now, by Lemma 3.11, $-r_1$ and $-r_1^{'}$ are clean. Then we may write $-r_1 = (1 - e_1) + (2e_1 - 1)$, where e_1 is an idempotent and $(2e_1 - 1)$ is a unit as $(2e_1 - 1)(2e_1 - 1) = 1$. So, $-r_1 = e_1$ is an idempotent. Similarly, $-r_2 = e_2$ is an idempotent. Then $a = e_1 + e_2 + (-n_1)$. Since all idempotents of R are orthogonal, therefore $e_1 + e_2$ is an idempotent and $(-n_1) \in N(R)$. Thus a is a sum of an idempotent and a nilpotent. Hence R is a nil clean ring.

$(2) \implies (3)$. It is clear from [6, Proposition 3.4].

$(3) \implies (4)$. It is clear from [11, Proposition 1.8].

$(4) \implies (1)$. Let R be an exchange ring. Then following [12], R is a clean ring as R is Abelian. Hence R is a 2-nil-regular ring by Remark 2.1(2). □

Remark 3.13. A 2-nil-regular ring need not be local. For example, let $R = \begin{pmatrix} \mathbb{Z}_4 & \mathbb{Z}_4 \\ 0 & \mathbb{Z}_4 \end{pmatrix}$. By Corollary 3.2, R is a 2-nil-regular ring but R is not a local ring.

In the following, we generalize [12, Lemma 14] as an application of Proposition 3.12.

Proposition 3.14. The following are equivalent for a nonzero ring R:

(i) R is a local ring.

(ii) R is a clean ring with the only idempotents 0 and 1.

(iii) R is a 2-nil-regular ring with the only idempotents 0 and 1.

Proof. $(1) \implies (2)$. Clear by [12, Lemma 14].

$(2) \implies (3)$. It is clear by Remark 2.1(2).

$(3) \implies (1)$. Let R be a 2-nil-regular ring. Since R has the only idempotents 0 and 1, therefore 0 and 1 are central and orthogonal. Then by Proposition 3.12, R is a clean ring and R is a local ring by [12, Lemma 14]. □

Recall by [8], a ring R is called (S, n)-ring if every element is a sum of no more than n units. A 2-nil regular ring need not be an $(S, 3)$-ring. For example, $\mathbb{Z}_4$ is a 2-nil regular ring but not an $(S, 3)$-ring as 2 is not written as sum of three units in $\mathbb{Z}_4$. We give a sufficient condition for a ring R to be $(S, 3)$-ring.

Proposition 3.15. Let R be a ring with the only idempotents 0 and 1. If R is a 2-nil-regular ring, then R is an $(S, 3)$-ring.

Proof. Suppose that R is a 2-nil-regular ring and let $a \in R$. We choose regular elements $0 \neq r_1$ and $0 \neq r_2$ and a nilpotent element n of R such that $a - 1 = r_1 + r_2 + n$. Since $r_1 \in reg(R)$, therefore there exists $y \in R$ such that $r_1 y r_1 = r_1$. Then $(r_1 y)^2 = r_1 y r_1 y = r_1 y$, also $(y r_1)^2 = y r_1 y r_1 = y r_1$. So $r_1 y$ and $y r_1$ are idempotents. Now, either $r_1 y = y r_1 = 1$ or at least one of $r_1 y$ and $y r_1$ is zero. Suppose at least one of $r_1 y$ and $y r_1$ is zero, then $r_1 = 0$ which is a contradiction. Similarly, we get $r_2 = 0$ which is again a contradiction. Suppose $r_1 y = y r_1 = 1$, then $r_1 \in U(R)$. Similarly, we get $r_2 \in U(R)$. Then from $a - 1 = r_1 + r_2 + n$, we have $a = r_1 + r_2 + (1 + n)$. So a is sum of three units. Hence R is an $(S, 3)$-ring. □

Proposition 3.16. Let $e_1, \ldots, e_n$ be orthogonal central idempotents whose sum is 1. Then R is 2-nil-regular if and only if each $e_i R e_i$ is 2-nil-regular.

Proof. Suppose R is a 2-nil-regular ring. Since $1 = e_1 + \ldots + e_n$, therefore $R = e_1 R + \ldots + e_n R$. Also, $R = e_1 R e_1 + \ldots + e_n R e_n$ as $e_1, \ldots, e_n$ are central. Since $e_1 R e_1, \ldots, e_n R e_n$ are subrings of R and $(e_i R e_i)(e_j R e_j) = 0$ for all i and j as $e_1, \ldots, e_n$ are orthogonal, therefore $R = e_1 R e_1 \oplus \ldots \oplus e_n R e_n$. Thus $e_1 R e_1, \ldots e_n R e_n$ are homomorphic images of R, so each $e_i R e_i$ is 2-nil-regular. Conversely, suppose that each $e_i R e_i$ is 2-nil-regular. Then the result follows from Corollary 3.3 and induction. □

Now, we have an analogous result in case of group ring.

Proposition 3.17. Let $e_1, \ldots e_n$ be orthogonal central idempotents whose sum is 1. Let G be any group. Then RG is 2-nil-regular if and only if each $(e_i R e_i)G$ is 2-nil-regular.

Proposition 3.18. Let R be a semiperfect ring with all idempotents central. Let G be a group and $(eRe)G$ be 2-nil-regular for each local idempotent e in R. Then RG is 2-nil-regular.

Proof. By [2, Theorem 27.6], R has a complete orthogonal set $\{e_1, \ldots, e_n\}$ of idempotents with each $e_i R e_i$ a local ring. So, $(e_i R e_i)G$ is 2-nil-regular. It follows by $(e_i R e_i)G \simeq e_i RG e_i$ for each i and Proposition 3.17 that $e_i RG e_i$ is 2-nil-regular. □

Proposition 3.19. Let R be a 2-nil-regular ring. Then $1_R = r_1 + r_2$ for some $r_1, r_2 \in reg(R)$. Moreover, the converse is true if R is a strongly π-regular ring with the only idempotents 0 and 1.

Proof. Let R be a 2-nil-regular ring. Then $1_R = r_3 + r_4 + n$, where $0 \neq r_3, 0 \neq r_4 \in reg(R)$ and $n \in N(R)$. Then $1 - n = r_3 + r_4$ and $1 = (1-n)(1-n)^{-1} = r_3(1-n)^{-1} + r_4(1-n)^{-1}$. Since $r_4 \in reg(R)$, therefore there exists $x \in R$ such that $r_4 x r_4 = r_4$. Now, $r_4(1-n)^{-1}((1-n)x)r_4(1-n)^{-1} = r_4 x r_4(1-n)^{-1} = r_4(1-n)^{-1}$. Similarly, $r_3(1-n)^{-1}((1-n)y)r_3(1-n)^{-1} = r_3(1-n)^{-1}$. Hence $1_R = r_1 + r_2$ for some $r_1, r_2 \in reg(R)$. Conversely, suppose that $1_R = r_1 + r_2$ for some $0 \neq r_1, 0 \neq r_2 \in reg(R)$. Since $r_1 \in reg(R)$, therefore there exists $y \in R$ such that $r_1 y r_1 = r_1$. Then $(r_1 y)^2 = r_1 y r_1 y = r_1 y$, also $(y r_1)^2 = y r_1 y r_1 = y r_1$. So $r_1 y$ and $y r_1$ are idempotents. Since R has the only 0 and 1 idempotents, therefore either $r_1 y = y r_1 = 1$ or at least one of $r_1 y$ and $y r_1$ is zero. Suppose at least one of $r_1 y$ and $y r_1$ is zero, then $r_1 = 0$ which is a contradiction. Similarly, we get $r_2 = 0$ which is also a contradiction. Suppose $r_1 y = y r_1 = 1$, then $r_1 \in U(R)$. Similarly, we get $r_2 \in U(R)$. Thus $1_R = u_1 + u_2$ for some u_1, $u_2 \in U(R)$. Then by [1, Theorem 2.1], R is a 2-nil-good ring as R is a strongly π-regular ring. Hence R is a 2-nil-regular ring. □

Corollary 3.20. Let R be a finite ring with the only idempotents 0 and 1. Then the following are equivalent:

(i) R is a 2-nil-regular ring.

(ii) $1_R = r_1 + r_2$ for some $r_1, r_2 \in reg(R)$.

Proof. Since R is a finite ring, therefore R is a strongly π-regular ring. Thus, the result follows from Proposition 3.19. □

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