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ONE PARAMETER FAMILY OF $\mathcal{B}$ - TANGENT DEVELOPABLE SURFACES OF SPACELIKE BIHARMONIC NEW TYPE $\mathcal{B}$-SLANT HELICES IN $\mathcal{H}^3$

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Abstract. In this paper, we study inextensible flows of tangent developable surfaces of biharmonic spacelike new type $\mathcal{B}$-slant helices according to Bishop frame in the Lorentzian Heisenberg group $\mathcal{H}^3$. We give necessary and sufficient conditions for new type $\mathcal{B}$-slant helices to be biharmonic. We characterize one parameter family of $\mathcal{B}$-tangent developable surfaces in the Lorentzian Heisenberg group $\mathcal{H}^3$. Additionally, we illustrate our results.

1 Introduction

A smooth map $\phi : N \longrightarrow M$ is said to be biharmonic if it is a critical point of the bienergy functional:

$$E_2(\phi) = \int_N \frac{1}{2} |\mathcal{T}(\phi)|^2 \, dv_h,$$

where $\mathcal{T}(\phi) := \mathrm{tr}\nabla^{\phi} d\phi$ is the tension field of ϕ

The Euler–Lagrange equation of the bienergy is given by $\mathcal{T}_2(\phi) = 0$. Here the section $\mathcal{T}_2(\phi)$ is defined by

$$\mathcal{T}_2(\phi) = -\Delta_\phi \mathcal{T}(\phi) + \mathrm{tr}R(\mathcal{T}(\phi), d\phi)\, d\phi,$$

and called the bitension field of ϕ. Non-harmonic biharmonic maps are called proper biharmonic maps.

This study is organised as follows: Firstly, we give necessary and sufficient conditions for new type $\mathcal{B}$-slant helices to be biharmonic. We characterize this curves in the Lorentzian Heisenberg group $\mathcal{H}^3$. Secondly, we study biharmonic $\mathcal{B}$-tangent developable surfaces of spacelike new type $\mathcal{B}$-slant helices according to Bishop frame in the Lorentzian Heisenberg group $\mathcal{H}^3$. Finally, we illustrate our results.

2 The Lorentzian Heisenberg Group $\mathcal{H}^3$

The Heisenberg group Heis^3 is a Lie group which is diffeomorphic to $\mathbb{R}^3$ and the group operation is defined as

$$(x, y, z) * (\overline{x}, \overline{y}, \overline{z}) = (x + \overline{x}, y + \overline{y}, z + \overline{z} - \overline{x}y + x\overline{y}).$$

The identity of the group is $(0, 0, 0)$ and the inverse of (x, y, z) is given by $(-x, -y, -z)$. The left-invariant Lorentz metric on Heis^3 is

$$g = -dx^2 + dy^2 + (xdy + dz)^2.$$

The following set of left-invariant vector fields forms an orthonormal basis for the corresponding Lie algebra:

$$\left\{ \mathbf{e}_1 = \frac{\partial}{\partial z},\ \mathbf{e}_2 = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z},\ \mathbf{e}_3 = \frac{\partial}{\partial x} \right\}. \tag{1}$$

The characterising properties of this algebra are the following commutation relations, [15]:

$$g(\mathbf{e}_1, \mathbf{e}_1) = g(\mathbf{e}_2, \mathbf{e}_2) = 1,\ \ g(\mathbf{e}_3, \mathbf{e}_3) = -1.$$

Proposition 2.1. *For the covariant derivatives of the Levi-Civita connection of the left-invariant metric g, defined above the following is true:*

$$\nabla = \frac{1}{2}\begin{pmatrix} 0 & \mathbf{e}_3 & \mathbf{e}_2 \\ \mathbf{e}_3 & 0 & \mathbf{e}_1 \\ \mathbf{e}_2 & -\mathbf{e}_1 & 0 \end{pmatrix}, \tag{2}$$

where the (i,j)-element in the table above equals $\nabla_{\mathbf{e}_i}\mathbf{e}_j$ for our basis

$$\{\mathbf{e}_k, k = 1, 2, 3\}.$$

3 Spacelike Biharmonic New Type $\mathcal{B}-$Slant Helices with Bishop Frame In The Lorentzian Heisenberg Group $\mathcal{H}^3$

Let $\gamma : I \longrightarrow \mathcal{H}^3$ be a non geodesic spacelike curve on the Lorentzian Heisenberg group $\mathcal{H}^3$ parametrized by arc length. Let $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ be the Frenet frame fields tangent to the Lorentzian Heisenberg group $\mathcal{H}^3$ along γ defined as follows:

$\mathbf{t}$ is the unit vector field γ' tangent to γ, $\mathbf{n}$ is the unit vector field in the direction of $\nabla_{\mathbf{t}}\mathbf{t}$ (normal to γ), and $\mathbf{b}$ is chosen so that $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ is a positively oriented orthonormal basis. Then, we have the following Frenet formulas:

$$\begin{aligned} \nabla_{\mathbf{t}}\mathbf{t} &= \kappa\mathbf{n}, \\ \nabla_{\mathbf{t}}\mathbf{n} &= \kappa\mathbf{t} + \tau\mathbf{b}, \\ \nabla_{\mathbf{T}}\mathbf{B} &= \tau\mathbf{n}, \end{aligned} \tag{3}$$

where κ is the curvature of γ and τ is its torsion and

$$g(\mathbf{t},\mathbf{t}) = 1,\ g(\mathbf{n},\mathbf{n}) = -1,\ g(\mathbf{b},\mathbf{b}) = 1,$$
$$g(\mathbf{t},\mathbf{n}) = g(\mathbf{t},\mathbf{b}) = g(\mathbf{n},\mathbf{b}) = 0.$$

In the rest of the paper, we suppose everywhere $\kappa \neq 0$ and $\tau \neq 0$.

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative. The Bishop frame is expressed as

$$\begin{aligned} \nabla_{\mathbf{t}}\mathbf{t} &= k_1\mathbf{m}_1 - k_2\mathbf{m}_2, \\ \nabla_{\mathbf{t}}\mathbf{m}_1 &= k_1\mathbf{t}, \\ \nabla_{\mathbf{t}}\mathbf{m}_2 &= k_2\mathbf{t}, \end{aligned} \tag{4}$$

where

$$g(\mathbf{t},\mathbf{t}) = 1,\ g(\mathbf{m}_1,\mathbf{m}_1) = -1,\ g(\mathbf{m}_2,\mathbf{m}_2) = 1,$$
$$g(\mathbf{T},\mathbf{M}_1) = g(\mathbf{t},\mathbf{m}_2) = g(\mathbf{m}_1,\mathbf{m}_2) = 0.$$

Here, we shall call the set $\{\mathbf{t}, \mathbf{m}_1, \mathbf{m}_2\}$ as Bishop trihedra, k_1 and k_2 as Bishop curvatures. Also, $\tau(s) = \psi'(s)$ and $\kappa(s) = \sqrt{|k_2^2 - k_1^2|}$. Thus, Bishop curvatures are defined by

$$\begin{aligned} k_1 &= \kappa(s)\sinh\psi(s), \\ k_2 &= \kappa(s)\cosh\psi(s). \end{aligned}$$

With respect to the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ we can write

$$\begin{aligned} \mathbf{t} &= t^1\mathbf{e}_1 + t^2\mathbf{e}_2 + t^3\mathbf{e}_3, \\ \mathbf{m}_1 &= m_1^1\mathbf{e}_1 + m_1^2\mathbf{e}_2 + m_1^3\mathbf{e}_3, \\ \mathbf{m}_2 &= m_2^1\mathbf{e}_1 + m_2^2\mathbf{e}_2 + m_2^3\mathbf{e}_3. \end{aligned} \tag{5}$$

Theorem 3.1. *$\gamma : I \longrightarrow \mathcal{H}^3$ is a spacelike biharmonic curve with Bishop frame if and only if*

$$\begin{aligned} k_1^2 - k_2^2 &= \text{constant} = C \neq 0, \\ k_1'' + \left[k_1^2 - k_2^2\right]k_1 &= -k_1\left[1 + \left(m_2^1\right)^2\right] + k_2 m_1^1 m_2^1, \\ k_2'' + \left[k_1^2 - k_2^2\right]k_2 &= -k_1 m_1^1 m_2^1 - k_2\left[-1 + \left(m_1^1\right)^2\right]. \end{aligned} \tag{6}$$

4 $\mathcal{B}-$Tangent Developable Surfaces of Spacelike Biharmonic New Type $\mathcal{B}-$Slant Helices with Bishop Frame In The Lorentzian Heisenberg Group $\mathcal{H}^3$

To separate a tangent developable according to Bishop frame from that of Frenet- Serret frame, in the rest of the paper, we shall use notation for this surface as $\mathcal{B}-$tangent developable.

The purpose of this section is to study $\mathcal{B}-$tangent developable of biharmonic spacelike new type $\mathcal{B}-$slant helix in $\mathcal{H}^3$. The $\mathcal{B}-$tangent developable of γ is a ruled surface

$$\mathcal{O}_{new}(s,u)=\gamma(s)+u\gamma'(s). \tag{7}$$

Definition 4.1. *A surface evolution $\mathcal{O}_{new}(s,u,t)$ and its flow $\dfrac{\partial\mathcal{O}_{new}}{\partial t}$ are said to be inextensible if its first fundamental form $\{\mathbf{E},\mathbf{F},\mathbf{G}\}$ satisfies*

$$\frac{\partial\mathbf{E}}{\partial t}=\frac{\partial\mathbf{F}}{\partial t}=\frac{\partial\mathbf{G}}{\partial t}=0. \tag{8}$$

Definition 4.2. *We can define the following one-parameter family of developable ruled surface*

$$\mathcal{O}_{new}(s,u,t)=\gamma(s,t)+u\gamma'(s,t). \tag{9}$$

Hence, we have the following theorem.

Theorem 4.3. *Let $\mathcal{O}_{new}$ be one-parameter family of the $\mathcal{B}-$tangent developable of a unit speed non-geodesic biharmonic new type $\mathcal{B}-$slant helix. Then $\dfrac{\partial\mathcal{O}_{new}}{\partial t}$ is inextensible if and only if*

$$\begin{aligned}
&\frac{\partial}{\partial t}(\sin\mathfrak{Q}(t)-uk_2(t)\cos\mathfrak{Q}(t))^2+\frac{\partial}{\partial t}(\cos\mathfrak{Q}(t)\cosh[\mathcal{C}_0(t)s+\mathcal{C}_1(t)]\\
&+uk_1(t)\sinh[\mathcal{C}_0(t)s+\mathcal{C}_1(t)]-uk_2(t)\sin\mathfrak{Q}(t)\cosh[\mathcal{C}_0(t)s+\mathcal{C}_1(t)])^2\\
&+\frac{\partial}{\partial t}(\cos\mathfrak{Q}(t)\sinh[\mathcal{C}_0(t)s+\mathcal{C}_1(t)]+uk_1(t)\cosh[\mathcal{C}_0(t)s+\mathcal{C}_1(t)]\\
&\qquad -uk_2(t)\sin\mathfrak{Q}(t)\sinh[\mathcal{C}_0(t)s+\mathcal{C}_1(t)])^2=0,
\end{aligned} \tag{10}$$

where $\mathcal{C}_0,\mathcal{C}_1$ are smooth functions of time.

Proof. Assume that $\mathcal{O}_{new}(s,u,t)$ be a one-parameter family of the $\mathcal{B}-$tangent developable of a unit speed non-geodesic biharmonic new type $\mathcal{B}-$slant helix.

From our assumption, we get the following equation

$$\mathbf{m}_2=\cos\mathfrak{Q}(t)\,\mathbf{e}_1+\sin\mathfrak{Q}(t)\cosh[\mathcal{C}_0(t)s+\mathcal{C}_1(t)]\mathbf{e}_2+\sin\mathfrak{Q}(t)\sinh[\mathcal{C}_0(t)s+\mathcal{C}_1(t)]\mathbf{e}_3. \tag{11}$$

where $\mathcal{C}_0,\mathcal{C}_1$ are smooth functions of time.

On the other hand, using Bishop formulas Eq.(4) and Eq.(1), we have

$$\mathbf{m}_1=\sinh[\mathcal{C}_0(t)s+\mathcal{C}_1(t)]\mathbf{e}_2+\cosh[\mathcal{C}_0(t)s+\mathcal{C}_1(t)]\mathbf{e}_3. \tag{12}$$

Using above equation and Eq.(11), we get

$$\mathbf{t}=\sin\mathfrak{Q}(t)\,\mathbf{e}_1+\cos\mathfrak{Q}(t)\cosh[\mathcal{C}_0(t)s+\mathcal{C}_1(t)]\mathbf{e}_2+\cos\mathfrak{Q}(t)\sinh[\mathcal{C}_0(t)s+\mathcal{C}_1(t)]\mathbf{e}_3. \tag{13}$$

Furthermore, we have the natural frame $\left\{(\mathcal{O}_{new})_s,(\mathcal{O}_{new})_u\right\}$ given by

$$(\mathcal{O}_{new})_s=(\sin\mathfrak{Q}(t)-uk_2(t)\cos\mathfrak{Q}(t))\mathbf{e}_1+(\cos\mathfrak{Q}(t)\cosh[\mathcal{C}_0(t)s+\mathcal{C}_1(t)]+ \tag{14}$$

$$uk_1(t)\sinh[\mathcal{C}_0(t)s+\mathcal{C}_1(t)]-uk_2(t)\sin\mathfrak{Q}(t)\cosh[\mathcal{C}_0(t)s+\mathcal{C}_1(t)])\mathbf{e}_2+$$
$$(\cos\mathfrak{Q}(t)\sinh[\mathcal{C}_0(t)s+\mathcal{C}_1(t)]+uk_1(t)\cosh[\mathcal{C}_0(t)s+\mathcal{C}_1(t)]-$$
$$uk_2(t)\sin\mathfrak{Q}(t)\sinh[\mathcal{C}_0(t)s+\mathcal{C}_1(t)])\mathbf{e}_3,$$

and

$$(\mathcal{O}_{new})_u=\sin\mathfrak{Q}(t)\,\mathbf{e}_1+\cos\mathfrak{Q}(t)\cosh[\mathcal{C}_0(t)s+\mathcal{C}_1(t)]\mathbf{e}_2+\cos\mathfrak{Q}(t)\sinh[\mathcal{C}_0(t)s+\mathcal{C}_1(t)]\mathbf{e}_3.$$

The components of the first fundamental form are

$$\begin{aligned}\frac{\partial\mathbf{E}}{\partial t} &= \frac{\partial}{\partial t}g((\mathcal{O}_{new})_s,(\mathcal{O}_{new})_s)=\frac{\partial}{\partial t}(\sin\mathfrak{Q}(t)-uk_2(t)\cos\mathfrak{Q}(t))^2\\ &+\frac{\partial}{\partial t}(\cos\mathfrak{Q}(t)\cosh[\mathcal{C}_0(t)s+\mathcal{C}_1(t)]+uk_1(t)\sinh[\mathcal{C}_0(t)s+\mathcal{C}_1(t)]\\ &-uk_2(t)\sin\mathfrak{Q}(t)\cosh[\mathcal{C}_0(t)s+\mathcal{C}_1(t)])^2\\ &+\frac{\partial}{\partial t}(\cos\mathfrak{Q}(t)\sinh[\mathcal{C}_0(t)s+\mathcal{C}_1(t)]+uk_1(t)\cosh[\mathcal{C}_0(t)s+\mathcal{C}_1(t)]\\ &-uk_2(t)\sin\mathfrak{Q}(t)\sinh[\mathcal{C}_0(t)s+\mathcal{C}_1(t)])^2,\end{aligned}$$

$$\frac{\partial\mathbf{F}}{\partial t}=0, \tag{15}$$

$$\frac{\partial\mathbf{G}}{\partial t} = 0.$$

Hence, $\dfrac{\partial\mathcal{O}_{new}}{\partial t}$ is inextensible if and only if Eq.(10) is satisfied. This concludes the proof of theorem.

Theorem 4.4. *Let $\mathcal{O}_{new}$ be one-parameter family of the $\mathcal{B}-$tangent developable surface of a unit speed non-geodesic biharmonic new type $\mathcal{B}-$slant helix. Then, the parametric equations of this family are given by*

$$\begin{aligned}\boldsymbol{x}_{\mathcal{O}_{new}}(s,u,t) &= \frac{1}{\mathcal{C}_0(t)}\cos\mathfrak{Q}(t)\cosh[\mathcal{C}_0(t)s+\mathcal{C}_1(t)]\\ &\quad+u\cos\mathfrak{Q}(t)\sinh[\mathcal{C}_0(t)s+\mathcal{C}_1(t)]+\mathcal{C}_2(t),\\ \boldsymbol{y}_{\mathcal{O}_{new}}(s,u,t) &= \frac{1}{\mathcal{C}_0(t)}\cos\mathfrak{Q}(t)\sinh[\mathcal{C}_0(t)s+\mathcal{C}_1(t)]\\ &\quad+u\cos\mathfrak{Q}(t)\cosh[\mathcal{C}_0(t)s+\mathcal{C}_1(t)]+\mathcal{C}_3(t),\end{aligned}$$

$$\boldsymbol{z}_{\mathcal{O}_{new}}(s,u,t)=\sin\mathfrak{Q}(t)\,s-\frac{\mathcal{C}_2(t)}{\mathcal{C}_0(t)}\cos\mathfrak{Q}(t)\sinh[\mathcal{C}_0(t)s+\mathcal{C}_1(t)] \tag{16}$$

$$-\frac{1}{4\mathcal{C}_0}\cos^2\mathfrak{Q}(\mathrm{t})\,(2[\mathcal{C}_0(t)s+\mathcal{C}_1(t)]$$

$$+\sinh 2[\mathcal{C}_0(t)s+\mathcal{C}_1(t)])+u\sin\mathfrak{Q}(t)-\mathrm{u}(\frac{1}{\mathcal{C}_0(t)}\cos\mathfrak{Q}(t)\cosh[\mathcal{C}_0(t)s+\mathcal{C}_1(t)]$$
$$+\mathcal{C}_2(t))\cos\mathfrak{Q}(t)\cosh[\mathcal{C}_0(t)s+\mathcal{C}_1(t)]+\mathcal{C}_4(t),$$

where $\mathcal{C}_1,\mathcal{C}_2,\mathcal{C}_3,\mathcal{C}_4,\mathcal{C}_0$ are smooth functions of time.

Proof. We assume that γ is a unit speed new type $\mathcal{B}-$slant helix.
Substituting Eq.(1) to Eq.(13), we have

$$\mathbf{t}=(\cos\mathfrak{Q}(t)\sinh[\mathcal{C}_0(t)s+\mathcal{C}_1(t)],\cos\mathfrak{Q}(t)\cosh[\mathcal{C}_0(t)s+\mathcal{C}_1(t)],\sin\mathfrak{Q}(t) \tag{17}$$

$$-(\frac{1}{\mathcal{C}_0(t)}\cos\mathfrak{Q}(t)\cosh[\mathcal{C}_0(t)s+\mathcal{C}_1(t)]+\mathcal{C}_2(t))\cos\mathfrak{Q}(t)\cosh[\mathcal{C}_0(t)s+\mathcal{C}_1(t)]),$$

Substituting this into the Eq.(15), we have Eq.(16). Thus, the proof is completed.

We can use Mathematica in above theorem, yields

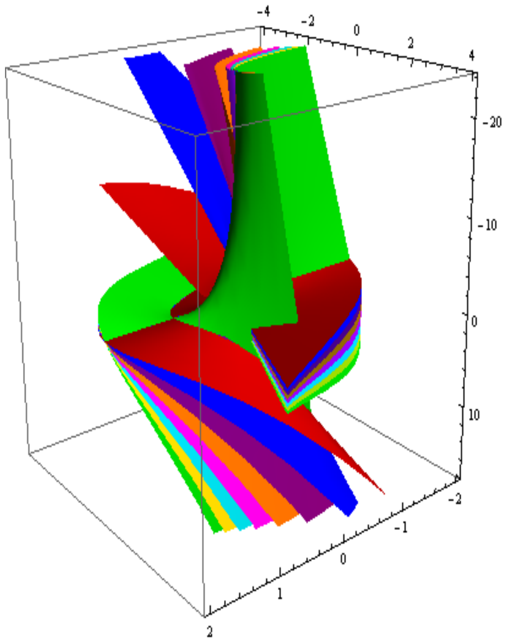

Figure 1.

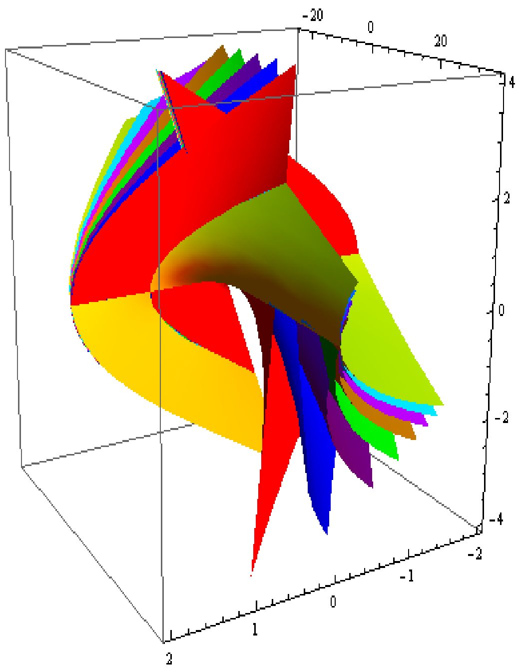

Figure 2.

Fig. 1,2: *The equation (16) is illustrated colour Red, Blue, Purple, Orange, Magenta, Cyan, Yellow, Green at the time* $t=1$, $t=1.2$, $t=1.4$, $t=1.6$, $t=1.8$, $t=2$, $t=2.2$, $t=2.4$, *respectively.*

References

[1] V. Asil, E. Turhan and T. Körpınar, *On characterization spacelike dual biharmonic curves in dual Lorentzian Heisenberg group $D^3_{\mathbf{Heis}^3}$*, Iranian Journal of Science & Tecnology Transaction A: Science (in press).

[2] M.do Carmo, *Differential Geometry of Curves and Surfaces*, Prentice Hall, New Jersey (1976).

[3] J. Eells and J. H. Sampson, *Harmonic mappings of Riemannian manifolds*, Amer. J. Math. 86, 109–160 (1964).

[4] R.T. Farouki, C.A. Neff, *Algebraic properties of plane offset curves*, Comput. Aided Geom. Design 7, 101–127 (1990).

[5] A. Gray, *Modern Differential Geometry of Curves and Surfaces with Mathematica*, CRC Press, (1998).

[6] G. Y.Jiang, *2-harmonic isometric immersions between Riemannian manifolds*, Chinese Ann. Math. Ser. A 7(2), 130–144 (1986).

[7] K. Onda, *Lorentz Ricci Solitons on 3-dimensional Lie groups*, Geom Dedicata 147 (1), 313-322 (2010).

[8] T. Körpınar, E. Turhan, *On characterization of B-canal surfaces in terms of biharmonic B-slant helices according to Bishop frame in Heisenberg group Heis^3*, J. Math. Anal. Appl. 382, 57–65 (2011).

[9] T. Körpınar, E. Turhan, *Spacelike biharmonic new type B-slant helices according to Bishop frame in Lorentzian Heisenberg group Heis^3*, Gen. Math. Notes 10 (2), 36-42 (2012).

[10] W. Lü, *Rationality of the offsets to algebraic curves and surfaces*. Appl. Math. (A Journal of Chinese Universities) 9 (Ser.B), 265 278 (1994).

[11] B. O'Neill, Semi-Riemannian Geometry, *Academic Press*, New York (1983).

[12] M. Peternell, H. Pottmann, *Computing rational parametrizations of canal surfaces*, J. Symb. Comput. 23 255–266 (1997).

[13] H. Pottmann, M. Peternell, *Applications of Laguerre geometry in CAGD*, Comput. Aided Geom. Design 15 165–186, (1998).

[14] U. Shani, D.H. Ballard, *Splines as embeddings for generalized cylinders*, Comput. Vision Graphics Image Process. 27, 129–156 (1984).

[15] E. Turhan, T. Körpınar, *On Characterization Of Timelike Horizontal Biharmonic Curves In The Lorentzian Heisenberg Group Heis^3*, Zeitschrift für Naturforschung A- A Journal of Physical Sciences 65a, 641-648 (2010).

[16] E. Turhan and T. Körpınar, *On Characterization Canal Surfaces around Timelike Horizontal Biharmonic Curves in Lorentzian Heisenberg Group Heis^3*, Zeitschrift für Naturforschung A- A Journal of Physical Sciences 66a, 441-449 (2011).

[17] L.Wang, M.C. Leu, D. Blackmore, *Generating sweep solids for NC verification using the SEDE method*, Proceedings of the Fourth Symposium on Solid Modeling and Applications, Atlanta, Georgia, 14–16 May 1997, pp. 364–375.

[18] Z. Xu, R. Feng and J. Sun, *Analytic and algebraic properties of canal surfaces*, Journal of Computational and Applied Mathematics, 195 (1-2), 220-228 (2006).

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