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On the study of unified representations of the generalized Voigt functions

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Abstract. In the present paper, we define a new representation for Voigt function involving the product of generalized Bessel function and Whittaker function. We first obtain its explicit representation and then we derive partly bilateral and partly unilateral representations for this function. Further, we establish some new and interesting generating functions involving various hypergeometric functions of one and more variables. Next, we give some summation formulae for this function. The results obtained for the generalized Voigt function defined by Srivastava et al. (1998) follow as special cases of our main findings.

1 Introduction

The well-known Voigt functions $K(x,y)$ and $L(x,y)$ occur frequently in a wide variety of physical problems such as astrophysical spectroscopy, neutron physics, plasma physics and statistical communication theory as well as some areas in mathematical physics and engineering associated with multidimensional analysis of spectral harmonics. Furthermore, the function $K(x,y)+iL(x,y)$ is, except to a numerical factor, identical to the so-called 'plasma dispersion function' which is tabulated by Fried and Conte [2]. Several generalizations, unifications and representations (integrals and series) of the Voigt functions have been given by a number of workers, for example, Srivastava and Miller [17, p.113, eq.(8)] introduced and studied systematically an unification (and generalization) of the Voigt functions $K(x,y)$ and $L(x,y)$ in the form

$$V_{\mu,\nu}\left(x,y\right)=\sqrt{\left(\frac{x}{2}\right)}\int_0^\infty t^\mu \exp\left(-yt-\frac{1}{4}t^2\right)\ J_\nu\left(xt\right)dt,\ \left(x,y\in R^+;Re\left(\mu+\nu\right)>-1\right) \tag{1.1}$$

where $J_\nu\left(z\right)$ is the well-known Bessel function of order ν and it can be easily seen that

$$V_{1/2,-1/2}\left(x,y\right)=K\left(x,y\right)\ \text{and}\ \mathrm{V}_{1/2,\,1/2}\left(\mathrm{x,y}\right)=\mathrm{L}\left(\mathrm{x,y}\right) \tag{1.2}$$

Subsequently, following the work of Srivastava and Miller [17] closely, Klusch [6] proposed a unification (and generalization) of the Voigt functions K(x, y) and L(x, y) in the form

$$\Omega_{\mu,\nu}\left(x,y,z\right)=\sqrt{\frac{x}{2}}\int_0^\infty t^\mu \exp\left(-yt-zt^2\right)\ J_\nu\left(xt\right)dt,\left(x,y,z\in R^+;Re\left(\mu+\nu\right)>-1\right) \tag{1.3}$$

In fact, it is easily verified by comparing (1.1) and (1.3) that

$$\Omega_{\mu,\nu}\left(2x\sqrt{z},2y\sqrt{z},z\right)=\left(2\sqrt{z}\right)^{-\mu-1/2}V_{\mu,\nu}\left(x,y\right) \tag{1.4}$$

and

$$\Omega_{\mu,\nu}\left(x,y,\frac{1}{4}\right)=V_{\mu,\nu}\left(x,y\right) \tag{1.5}$$

Srivastava and Chen [14], Srivastava et al. [18], Pathan and Shahwan [9], Goyal and Mukherjee [3], Gupta et al. [5] and Pathan et al. [8] have also introduced and studied generalization of Voigt functions.

The main object of the present paper is to introduce and investigate a new generalized Voigt function involving the product of generalized Bessel function and Whittaker function defined as follows

$$\Omega_{\eta,\nu,\lambda}^{\mu,\rho,\sigma}(x,y,z,u)=\sqrt{\frac{x}{2}}\int_0^\infty t^{\eta+2\rho}\ e^{-yt-zt^2}\ J_{\nu,\lambda}^{\mu}(xt)\ M_{\rho,\sigma}\left(2ut^2\right)\ dt \tag{1.6}$$

$$\left(x,y,z,u\in R^+, Re\left[\eta+\nu+2\left(\rho+\sigma+\lambda\right)\right]>-2\right)$$

where $J_{\nu,\lambda}^{\mu}(z)$ is a generalization of Bessel function defined by Pathak [7] as follows

$$J_{\nu,\lambda}^{\mu}(z)=\sum_{m=0}^{\infty}\frac{(-1)^m\left(z/2\right)^{\nu+2\lambda+2m}}{\Gamma(\lambda+m+1)\,\Gamma(\nu+\lambda+\mu m+1)} \tag{1.7}$$

and $M_{\rho,\sigma}(z)$ is the Whittaker function [15, p.243, eq.(A.9)]
In (1.6), if we take $\sigma=-\frac{1}{2}-\rho$ and $u=\frac{z}{2}$, we get the generalized Voigt function given by Srivastava et al. [18]. Which gives the known generalization of Voigt function defined by Srivastava and Chen [14] on taking $z=1/2$.

2 Explicit representation for the generalized Voigt function

To obtain the explicit representation for the generalized Voigt function, we express the generalized Bessel function $J_{\nu,\lambda}^{\mu}(xt)$, the exponential function exp(-yt) and Whittaker function $M_{\rho,\sigma}\left(2ut^2\right)$ in their series forms and interchanging the order of summation and integration (which is permissible under the conditions stated), we get

$$\begin{aligned}\Omega_{\eta,\nu,\lambda}^{\mu,\rho,\sigma}(x,y,z,u)&=\left(\tfrac{x}{2}\right)^{\nu+2\lambda+1/2}(2u)^{\frac{1}{2}+\sigma}\sum_{s=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\tfrac{(-1)^m\left(\frac{1}{2}+\sigma-\rho\right)_n(x/2)^{2m}}{\Gamma(\lambda+m+1)\Gamma(\nu+\lambda+\mu m+1)}\\ &\times\tfrac{(-y)^s(2u)^n}{(2\sigma+1)_n s!n!}\int_0^\infty t^{\eta+2\rho+s+\nu+2\sigma+2\lambda+2m+2n+1}\ e^{-(z+u)t^2}dt\end{aligned} \tag{2.1}$$

Now evaluating the above t- integral with the help of the integral

$$\int_0^\infty t^\lambda e^{-zt^2}dt=\frac{1}{2}\Gamma\left(\frac{\lambda+1}{2}\right)(z)^{-(\lambda+1)/2},\quad Re\,(z)>0,\ Re\,(\lambda)>-1 \tag{2.2}$$

and doing some simplifications we arrive at the following result

$$\begin{aligned}\Omega_{\eta,\nu,\lambda}^{\mu,\rho,\sigma}(x,y,z,u)&=\tfrac{2^{\sigma-\nu-2\lambda-1}\,x^{\nu+2\lambda+1/2}\,u^{\sigma+1/2}}{Z^\alpha}\sum_{s,m,n=0}^{\infty}\tfrac{\left(\frac{1}{2}+\sigma-\rho\right)_n\Gamma\left(m+n+\alpha+\frac{s}{2}\right)}{\Gamma(\lambda+m+1)\Gamma(\nu+\lambda+\mu m+1)}\\ &\times\tfrac{1}{(2\sigma+1)_n s!n!}\left(-\tfrac{x^2}{4Z}\right)^m\left(-\tfrac{y}{\sqrt{Z}}\right)^s\left(\tfrac{2u}{Z}\right)^n\end{aligned} \tag{2.3}$$

where we have used the symbols Z and α for Z = z+u and $\alpha=1+\rho+\sigma+\lambda+\frac{\eta+\nu}{2}$.
On separating the s-series into its even and odd terms, we obtain

$$\begin{aligned}\Omega_{\eta,\nu,\lambda}^{\mu,\rho,\sigma}(x,y,z,u)&=\tfrac{2^{\sigma-\nu-2\lambda-1}x^{\nu+2\lambda+1/2}u^{\sigma+1/2}}{Z^\alpha}\sum_{m=0}^{\infty}\tfrac{\left(-x^2/4z\right)^m}{\Gamma(\lambda+m+1)\Gamma(\nu+\lambda+\mu m+1)}\\ &\times\Big\{\Gamma(\alpha+m)\,\psi_1\left(\alpha+m,\tfrac{1}{2}+\sigma-\rho;2\sigma+1,\tfrac{1}{2};\tfrac{2u}{Z},\tfrac{y^2}{4Z}\right)-\tfrac{y}{\sqrt{Z}}\Gamma\left(\alpha+\tfrac{1}{2}+m\right)\\ &\times\psi_1\left(\alpha+\tfrac{1}{2}+m,\tfrac{1}{2}+\sigma-\rho;2\sigma+1,\tfrac{3}{2};\tfrac{2u}{Z},\tfrac{y^2}{4Z}\right)\end{aligned} \tag{2.4}$$

$$\left(x,y,z,u\in R^+,\ \mathrm{Re}\left(\eta+\nu+2\left(\rho+\sigma+\lambda\right)\right)>-2\right)$$

where ψ_1 denotes one of the Humbert's confluent hypergeometric function of two variables defined by Srivastava and Manocha [16, p.59, eq. 1.6(41)]

$$\psi_1\left[\alpha,\beta\,;\gamma,\gamma';x,y\right]=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m+n}\,(\beta)_m}{(\gamma)_m\,(\gamma')_n}\frac{x^m}{m!}\frac{y^n}{n!},\quad\left(|x|<1\,;|y|<\infty\right) \tag{2.5}$$

If we put $\sigma=-\frac{1}{2}-\rho$ and u = z/2 in equation (2.3) and (2.4), we get the explicit form of generalized Voigt function obtained by Srivastava et al. [18, p.54, eq.(2.3) and p.55, eq.(2.4)].

3 Partly Bilateral and Partly Unilateral Representations

Result 1

We start with the following result given by Srivastava, Bin-Saad and Pathan [13, p.8, eq.(1.3)]

$$\exp\left[s+t-\frac{xt}{s}\right]=\sum_{m=-\infty}^{\infty}\sum_{p=0}^{\infty}\frac{s^m}{m!}\frac{t^p}{p!}\,{}_1F_1\left[-p;m+1;x\right] \tag{3.1}$$

where ${}_1F_1\left(a;c;z\right)$ is the confluent hypergeometric function.
Replacing $s\to s\xi^2, t\to t\xi^2, x\to x\xi^2$ in (3.1) and multiplying both the sides by

$$\xi^{\eta+2\rho}\exp\left[-w\xi-z\xi^2\right]J^{\mu}_{\nu,\lambda}\left(q\xi\right)M_{\rho,\sigma}\left(2u\xi^2\right)$$

and integrating with respect to ξ from 0 to ∞, we obtain

$$\begin{aligned}&\int_0^{\infty}\xi^{\eta+2\rho}\exp\left[-w\xi-\left\{z-\left(s+t-\tfrac{xt}{s}\right)\right\}\xi^2\right]J^{\mu}_{\nu,\lambda}\left(q\xi\right)M_{\rho,\sigma}\left(2u\xi^2\right)d\xi\\&=\sum_{m=-\infty}^{\infty}\sum_{p=0}^{\infty}\tfrac{s^m}{m!}\tfrac{t^p}{p!}\int_0^{\infty}\xi^{\eta+2[\rho+m+p]}\\&\times\exp\left[-w\xi-z\xi^2\right]J^{\mu}_{\nu,\lambda}\left(q\xi\right)M_{\rho,\sigma}\left(2u\xi^2\right){}_1F_1\left[-p;m+1;x\xi^2\right]d\xi\end{aligned} \tag{3.2}$$

On comparing the equation (3.2) with (1.6), we obtain the following expression

$$\begin{aligned}&\Omega^{\mu,\rho,\sigma}_{\eta,\nu,\lambda}\left(q,w,z-s-t+\tfrac{xt}{s},u\right)=\sqrt{\tfrac{q}{2}}\sum_{m=-\infty}^{\infty}\sum_{p=0}^{\infty}\tfrac{s^m}{m!}\tfrac{t^p}{p!}\int_0^{\infty}\xi^{\eta+2[\rho+m+p]}\\&\quad\times\exp\left[-w\xi-z\xi^2\right]J^{\mu}_{\nu,\lambda}\left(q\xi\right)M_{\rho,\sigma}\left(2u\xi^2\right)_1F_1\left[-p;m+1;x\xi^2\right]d\xi\\&\left[q,w,z,u\in R^+,\ \left(z-s-t+xt/s\right)\in R^+,Re\left(\eta+\nu+2\left(\rho+\sigma+\lambda\right)>-2\right)\right]\end{aligned} \tag{3.3}$$

Now expanding the exponential functionexp $\left(-w\xi\right)$, generalized Bessel function $J^{\mu}_{\nu,\lambda}\left(q\xi\right)$and Whittaker function $M_{\rho,\sigma}\left(2u\xi^2\right)$in series form and using the known result [1, p.337, eq.(2.2)]

$$\begin{aligned}&\int_0^{\infty}x^{s-1}\exp\left(-\alpha x^2\right)_1F_1\left[a;b;\beta x^2\right]dx=\tfrac{\alpha^{-s/2}}{2}\Gamma\left(\tfrac{s}{2}\right)_2F_1\left[a,\tfrac{s}{2};b;\tfrac{\beta}{\alpha}\right]\\&\qquad Re\left(s\right)>0,\ Re\left(\alpha\right)>\max\left\{0,Re\left(\beta\right)\right\}\end{aligned} \tag{3.4}$$

we obtain

$$\begin{aligned}&\Omega^{\mu,\rho,\sigma}_{\eta,\nu,\lambda}\left(q,w,z-s-t+\tfrac{xt}{s},u\right)=\tfrac{2^{\sigma-\nu-2\lambda-1}q^{\nu+2\lambda+1/2}u^{\sigma+1/2}}{Z^{\alpha}}\sum_{m=-\infty}^{\infty}\sum_{p=0}^{\infty}\tfrac{(s/Z)^m}{m!}\tfrac{(t/Z)^p}{p!}\\&\times\sum_{j,r,n=0}^{\infty}\tfrac{\left(\frac{1}{2}+\sigma-\rho\right)_n\Gamma\left(\alpha+m+p+n+j+\frac{r}{2}\right)}{\Gamma(\lambda+j+1)\Gamma(\nu+\lambda+\mu j+1)(2\sigma+1)_n r!n!}\\&\cdot{}_2F_1\left[\begin{matrix}-p,m+p+\alpha+n+j+\frac{r}{2}; & \\ m+1 & ;\end{matrix}\left(\tfrac{x}{Z}\right)\right]\left(-\tfrac{w}{\sqrt{Z}}\right)^r\left(-\tfrac{q^2}{4Z}\right)^j\left(\tfrac{2u}{Z}\right)^n\end{aligned} \tag{3.5}$$

Now writing ${}_2F_1$ in series form and separating the r-series into its even and odd terms, we get

$$\begin{aligned}&\Omega^{\mu,\rho,\sigma}_{\eta,\nu,\lambda}\left(q,w,z-s-t+\tfrac{xt}{s},u\right)=\tfrac{2^{\sigma-\nu-2\lambda-1}q^{\nu+2\lambda+1/2}u^{\sigma+1/2}}{Z^{\alpha}}\sum_{m=-\infty}^{\infty}\sum_{p=0}^{\infty}\tfrac{(s/Z)^m}{m!}\tfrac{(t/Z)^p}{p!}\\&\times\sum_{j=0}^{\infty}\tfrac{\left(-q^2/4Z\right)^j}{\Gamma(\lambda+j+1)\Gamma(\nu+\lambda+\mu j+1)}\left[\Gamma\left(\alpha+m+p+j\right)\right.\\&\times F^3\left\{\begin{matrix}\alpha+m+p+j::-;-;-;\ -p\ ;\left(\frac{1}{2}+\sigma-\rho\right);\ -\ ; & \\ -::-;-;-;m+1;\quad 2\sigma+1\quad;1/2; & \left(\frac{x}{Z}\right),\left(\frac{2u}{Z}\right),\left(\frac{w^2}{4Z}\right)\end{matrix}\right\}\\&-\tfrac{w}{\sqrt{Z}}\Gamma\left(\alpha+m+p+j+\tfrac{1}{2}\right)\\&\left.\times F^3\left\{\begin{matrix}\alpha+m+p+j+1/2::-;-;-;\ -p\ ;\left(\frac{1}{2}+\sigma-\rho\right);\ -\ ; & \\ -::-;-;-;m+1;\quad 2\sigma+1\quad;3/2; & \left(\frac{x}{Z}\right),\left(\frac{2u}{Z}\right),\left(\frac{w^2}{4Z}\right)\end{matrix}\right\}\right]\\&\qquad\left[q,w,z,u\in R^+,\ \left(z-s-t+xt/s\right)\in R^+,Re\left(\eta+\nu+2\left(\rho+\sigma+\lambda\right)>-2\right)\right]\end{aligned} \tag{3.6}$$

where $F^3\left[x,y,z\right]$ denotes Srivastava's triple hypergeometric series defined in Srivastava and Manocha [16, p.69, eq. 1.7(39)]

$$F^{(3)}\left[\begin{array}{l}(a)::(b);(b');(b''):(c);(c');(c'');\\(e)::(g);(g');(g''):(h);(h');(h'');\end{array}x,y,z\right]$$
$$=\sum_{m,n,p=0}^{\infty}\frac{(a)_{m+n+p}(b)_{m+n}(b')_{n+p}(b'')_{p+m}(c)_m(c')_n(c'')_p}{(e)_{m+n+p}(g)_{m+n}(g')_{n+p}(g'')_{p+m}(h)_m(h')_n(h'')_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!} \tag{3.7}$$

4 Generating Functions

We start with the result (3.6) and express its left hand side using (2.4), we obtain the following generating function

$$\left(\frac{Z-s-t+xt/s}{Z}\right)^{-\alpha}\sum_{j=0}^{\infty}\frac{\left(-\frac{q^2}{4(Z-s-t+xt/s)}\right)^j}{\Gamma(\lambda+j+1)\Gamma(\nu+\lambda+\mu j+1)}$$
$$\times\left\{\Gamma(\alpha+j)\,\psi_1\left(\alpha+j,\tfrac{1}{2}+\sigma-\rho;2\sigma+1,\tfrac{1}{2};\frac{2u}{\left(Z-s-t+\frac{xt}{s}\right)},\frac{w^2}{4\left(Z-s-t+\frac{xt}{s}\right)}\right)-\frac{w}{\sqrt{\left(Z-s-t+\frac{xt}{s}\right)}}\right.$$
$$\left.\times\Gamma\left(\alpha+\tfrac{1}{2}+j\right)\psi_1\left(\alpha+\tfrac{1}{2}+j,\tfrac{1}{2}+\sigma-\rho;2\sigma+1,\tfrac{3}{2};\frac{2u}{\left(Z-s-t+\frac{xt}{s}\right)},\frac{w^2}{4\left(Z-s-t+\frac{xt}{s}\right)}\right)\right\}$$
$$=\sum_{m=-\infty}^{\infty}\sum_{p=0}^{\infty}\frac{(s/Z)^m}{m!}\frac{(t/Z)^p}{p!}\sum_{j=0}^{\infty}\frac{\left(-q^2/4Z\right)^j}{\Gamma(\lambda+j+1)\Gamma(\nu+\lambda+\mu j+1)}\left[\Gamma(\alpha+m+p+j)\right.$$
$$\times F^3\left\{\begin{array}{l}\alpha+m+p+j::-;-;-;\ -p\ ;\left(\tfrac{1}{2}+\sigma-\rho\right);\ -\ ;\\ \qquad -::-;-;-;m+1;\ \ 2\sigma+1\ \ ;1/2;\end{array}\left(\tfrac{x}{Z}\right),\left(\tfrac{2u}{Z}\right),\left(\tfrac{w^2}{4Z}\right)\right\}-$$
$$\frac{w}{\sqrt{Z}}\Gamma\left(\alpha+m+p+j+\tfrac{1}{2}\right)$$
$$\left.\times F^3\left\{\begin{array}{l}\alpha+m+p+j+1/2::-;-;-;\ -p\ ;\left(\tfrac{1}{2}+\sigma-\rho\right);\ -\ ;\\ \qquad -::-;-;-;m+1;\ \ 2\sigma+1\ \ ;3/2;\end{array}\left(\tfrac{x}{Z}\right),\left(\tfrac{2u}{Z}\right),\left(\tfrac{w^2}{4Z}\right)\right\}\right] \tag{4.1}$$

On taking $w\to 0$ in the above relation, we get the following generating function
Generating function-1

$$\left(\frac{Z-s-t+xt/s}{Z}\right)^{-\alpha}\sum_{j=0}^{\infty}\frac{\left(-\frac{q^2}{4(Z-s-t+xt/s)}\right)^j}{\Gamma(\lambda+j+1)\Gamma(\nu+\lambda+\mu j+1)}$$
$$\times\Gamma(\alpha+j)\ {}_2F_1\left(\begin{array}{c}\alpha+j\ ,\ \tfrac{1}{2}+\sigma-\rho\\ 2\sigma+1\end{array};\frac{2u}{(Z-s-t+xt/s)}\right)$$
$$=\sum_{m=-\infty}^{\infty}\sum_{p=0}^{\infty}\frac{(s/Z)^m}{m!}\frac{(t/Z)^p}{p!}\sum_{j=0}^{\infty}\frac{\left(-q^2/4Z\right)^j}{\Gamma(\lambda+j+1)\Gamma(\nu+\lambda+\mu j+1)}\left[\Gamma(\alpha+m+p+j)\right.$$
$$\times F_2\left[\alpha+m+p+j,\ -p,\ \tfrac{1}{2}+\sigma-\rho\,;m+1,\ 2\sigma+1\,;\tfrac{x}{Z},\tfrac{2u}{Z}\right] \tag{4.2}$$

If we take $\sigma=-\frac{1}{2}-\rho$, let $u\to 0$ and $q\to 0$ in the equation (4.1), we arrive at a known result obtained by Srivastava et al. [18, p.62, eq.(4.2)]. Further if we set $w\to 0$ and use the known relation [12, p.254, eq.1], we arrive at known result given by Pathan and Yasmeen [10, p.242, eq.(2.2)].

5 Connection between $V_{\mu,\nu}$ and $\Omega_{\mu,\alpha,\beta,\nu}$

$$(i)\sum_{k=0}^{\infty}\frac{(\mu)_k}{(\lambda)_k\,k!}\left(\frac{x}{2}\right)^k V_{\mu+k,k+\nu}(x,y,z)=\frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^{\nu}\Omega_{\mu+\nu,\lambda-\mu,\lambda,\nu}(x,y,z) \tag{5.1}$$

$$(ii)\sum_{k=0}^{\infty}\frac{(2-\nu)_k}{(\nu)_k}\ V_{\mu,2k+1}(x,y,z)=\frac{x}{2}\Omega_{\mu+1,\nu-1/2,3/2,\nu-1}(x,y,z) \tag{5.2}$$

Proof: To establish the relation (5.1), we consider the known result [11, p.662, eq.5.7.8.(3)]

$$\sum_{k=0}^{\infty}\frac{(\mu)_k}{(\lambda)_k\,k!}\left(\frac{x}{2}\right)^k J_{k+\nu}(x)=\frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^{\nu}{}_1F_2\left(\lambda-\mu;\lambda,1+\nu;-\frac{x^2}{4}\right) \tag{5.3}$$

Now replacing x by xt, multiplying both sides of the above relation by $t^{\mu}e^{-yt-zt^2}$ and integrating with respect to t from 0 to ∞, we obtain

$$\begin{aligned}&\sum_{k=0}^{\infty}\tfrac{(\mu)_k}{(\lambda)_k k!}\left(\tfrac{x}{2}\right)^k\int_0^{\infty}t^{\mu+k}e^{-yt-zt^2}J_{k+\nu}(xt)\,dt\\&=\tfrac{1}{\Gamma(\nu+1)}\left(\tfrac{x}{2}\right)^{\nu}\int_0^{\infty}t^{\mu+\nu}e^{-yt-zt^2}{}_1F_2\left(\lambda-\mu;\lambda,1+\nu;-\tfrac{x^2t^2}{4}\right)dt\end{aligned}\tag{5.4}$$

Interpreting the integral occurring in the above result by using the generalized Voigt function given by Pathan and Shahwan [9], we arrive at the desired result.
The other relation can be easily derived in a similar manner by using a known result [11, p.658, eq.9].

6 Summation Formulae

In this section we shall obtain three summation formulae involving the Voigt function.
Formula-1

$$\begin{aligned}&\sum_{k=0}^{\infty}\tfrac{(4k+1)(2k-1)!!}{2^k k!}V_{\mu,2k+1/2}(x,y)\\&=x\Gamma\left(\mu+\tfrac{3}{2}\right)\left\{\tfrac{1}{\Gamma\left(\frac{\mu}{2}+\frac{5}{4}\right)}{}_1F_1\left[\tfrac{\mu}{2}+\tfrac{3}{4};\tfrac{1}{2};y^2\right]-\tfrac{2y}{\Gamma\left(\frac{\mu}{2}+\frac{3}{4}\right)}{}_1F_1\left[\tfrac{\mu}{2}+\tfrac{5}{4};\tfrac{3}{2};y^2\right]\right\}\end{aligned}\tag{6.1}$$

where (2k-1)!! = 1.3.....(2k-1) and (-1)!! = 1 (see e.g. [4, p. (XLiii)]).
Formula-2

$$\sum_{k=0}^{\infty}\frac{(x/2)^k}{(\nu+k)\,k!}V_{\mu+k,k-m}(x,y)=(-1)^m\Gamma(\nu)\left(\frac{2}{x}\right)^{\nu}V_{\mu-\nu,m+\nu}(x,y)\tag{6.2}$$

Formula-3

$$\sum_{k=0}^{\infty}\frac{(-1)^k(k+\nu)(2\nu)_k}{k!}V_{\mu,\nu+k,\nu+k}(x,w,y)=\frac{1}{\Gamma(\nu)}\left(\frac{wx}{2(w+x)}\right)^{\nu}V_{\mu+\nu,\nu}(w+x,y)\tag{6.3}$$

Derivation of formula-1: To establish the formula-1, we use the following known result [4, p.934, eq. 8.512(3)]

$$\sum_{k=0}^{\infty}\frac{(4k+1)\,(2k-1)!!}{2^k\,k!}J_{2k+1/2}(x)=\sqrt{\frac{2}{\pi}x}\tag{6.4}$$

Now replacing x by xt, multiplying both sides of the above equation by $t^{\mu}\,e^{-yt-\frac{1}{4}t^2}$ and integrating with respect to t from 0 to∞ and using the definition (1.1), we obtain

$$\sum_{k=0}^{\infty}\frac{(4k+1)(2k-1)!!}{2^k\,k!}V_{\mu,2k+1/2}(x,y)=\frac{x}{\sqrt{\pi}}\int_0^{\infty}t^{\mu+1/2}\,e^{-yt-\frac{1}{4}t^2}dt\tag{6.5}$$

Finally, evaluating the t-integral by using the following integral representation [6, p.231, eq.(12)].

$$\begin{aligned}&\int_0^{\infty}t^{\sigma-1}\exp\left(-yt-\tfrac{t^2}{4}\right)dt=(2)^{\sigma/2}\Gamma(\sigma)\exp\left(\tfrac{y^2}{2}\right)D_{-\sigma}\left(\sqrt{2}y\right)\quad(y\in C;\mathrm{Re}(\sigma)>0)\\&=\Gamma(1/2)\Gamma(\sigma)\left\{\tfrac{1}{\Gamma\left(\frac{1+\sigma}{2}\right)}{}_1F_1\left[\begin{matrix}\sigma/2;\\1/2;\end{matrix}\;y^2\right]-\tfrac{2y}{\Gamma(\sigma/2)}{}_1F_1\left[\begin{matrix}\frac{1+\sigma}{2};\\3/2;\end{matrix}\;y^2\right]\right\}\end{aligned}\tag{6.6}$$

where $D_{-\sigma}(x)$ is parabolic cylinder function [16].

In the right hand side of the equation (6.5), we arrive at the required result after simplification. The formula 2 and 3 can be developed on the similar lines by using the following known results recorded in the well known work by Prudnikov et al. [11, p.661, eq.(6) ; p.671, eq.(13)]

$$(i)\sum_{k=0}^{\infty}\frac{(x/2)^k}{(\nu+k)\,k!}J_{k-m}(x)=(-1)^m\Gamma(\nu)\left(\frac{2}{x}\right)^{\nu}J_{m+\nu}(x)\tag{6.7}$$

and

$$(ii)\sum_{k=0}^{\infty}\frac{(-1)^k(k+\nu)(2\nu)_k}{k!}J_{\nu+k}(w)\,J_{\nu+k}(x)=\frac{1}{\Gamma(\nu)}\left(\frac{wx}{2(w+x)}\right)^{\nu}J_{\nu}(w+x)\tag{6.8}$$

References

[1] Erdélyi, A., Magnus, W., Oberhettinger, F. and Tricomi, F.G.: Tables of Integral Transforms, Vol.1, McGraw-Hill, New York (1954).

[2] Fried, B.D. and Conte, S.D.: The Plasma Dispersion Function, Academic Press, New York, (1961).

[3] Goyal, S.P. and Mukherjee, R.: Generalizations of the Voigt functions through generalized Lauricella function, Ganita Sandesh, **13** (1), 31-41 (1999).

[4] Gradshteyn, I.S. and Ryzhik, I.M., Table of Integrals, Series and Products, Academic Press, (2007).

[5] Gupta, K.C., Goyal, S.P. and Mukherjee, R.: Some results on generalized Voigt functions, ANZIAM, **44** , 299-303 (2002).

[6] Klusch, D.: Astrophysical spectroscopy and neutron reactions: Integral transforms and Voigt functions, Astrophys. Space Sci., **175** , 229-240 (1991).

[7] Pathak, R.S.: Certain convergence theorems and asymptotic properties of a generalization of Lommel and Maitland transform, Proc. U.S. Nat. Acad. Sci. Sect. A., **36**, 81-86 (1966).

[8] Pathan, M.A., Gupta, K. and Agrawal, V.: Summation formulae involving Voigt functions and generalized hypergeometric functions, SCIENTIA Series: A Mathematical Sciences, **19**, 37-44 (2010).

[9] Pathan, M.A. and Shahwan, M.J.S.: New representations of the Voigt functions, Demonstratio Math., **39**, 1-4 (2006).

[10] Pathan, M.A. and Yasmeen: On partly bilateral and partly unilateral generating functions, J. Austral. Math. Soc., Ser. B **28**, 240-245 (1986).

[11] Prudnikov, A.P., Brychkov Yu., A. and Marichev, O.I.: Integrals and Series, Vol. II, Gordan and Breach Science Publishers, New York (1986).

[12] Rainville, E.D.: Special Functions, Chelsea Publishing Company, New York (1960).

[13] Srivastava, H.M., Bin-Saad, M.G. and Pathan, M.A.: A new theorem on multidimensional generating relations and its applications, Proceeding of the Jangjeon Mathematical Society, No.1, **10**, 7-22 (2007).

[14] Srivastava, H.M. and Chen, M.-P.: Some unified presentations of the Voigt functions, Astrophys. Space Sci., **192**, 63-74 (1992).

[15] Srivastava, H.M., Gupta, K.C. and Goyal, S.P.: The H-function of One and Two variables with Applications, South Asian Publishers, New Delhi and Madras (1982).

[16] Srivastava, H.M. and Manocha, H.L.: A Treatise on Generating Functions, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York (1984).

[17] Srivastava, H.M. and Miller, E.A.: A unified presentation of the Voigt functions, Astrophys. Space Sci., **135**, 111-118 (1987).

[18] Srivastava, H.M., Pathan, M.A. and Kamarujjama, M.: Some unified presentations of the generalized Voigt functions, Commun. Appl. Anal., **2** , 49-64 (1998).

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