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Transformation of 4-D dynamical systems to hyperjerk form

Zeraoulia Elhadj and J. C. Sprott

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Abstract. In this paper, we investigate the possible cases for equivalence between four-dimensional autonomous dynamical systems and hyperjerk dynamics. In fact, we show that a wide class of four-dimensional vector fields possess this property.

1 Introduction

A jerky dynamical system is a differential equation of the form $x''' = J\left(x, x', x''\right)$ with an explicit time dependence (see [1] and [5]). The first three derivatives $\left(x', x'', x'''\right)$ are called *velocity*, *acceleration*, and *jerk*, respectively. Jerky dynamics form an interesting subclass of dynamical systems that can exhibit regular and chaotic behavior such as shown by several simple examples in [6-7-8-9]. Jerky dynamics can be found in several nonmechanical areas of physics, one example of which is the chaotic third-order differential equation representing an oscillator model of thermal convection given in [3]. In fact, jerky dynamics are conceptually simpler than dynamical systems, and they show all the major features of three-dimensional vector fields. In particular, they can be considered as the natural generalization of oscillator dynamics, and hence new methods to study chaos are possible.

By extension, a hyperjerk system is a dynamical system governed by an n^{th}-order ordinary differential equation with $n > 3$ describing the time evolution of a single scalar variable of the form $x^{(4)} = H\left(x, x', x'', x''', ...\right)$. The fourth derivative $x^{(n)} = x''''$ has been called a *snap*, with successive derivatives *crackle* and *pop*, and there is no universally accepted name for the higher derivatives. Some simple forms of 4^{th} and 5^{th} order displaying chaos and hyperchaos were presented in [8]. The connection between externally driven nonlinear oscillators and specific uni- and bi-directionally coupled systems of two autonomous oscillators was discussed in [4] along with a reinterpretation of simple chaotic forms of hyperjerk systems with some criteria that exclude chaotic behavior in some classes of these hyperjerk systems. In fact, some 4-D quadratic continuous-time autonomous systems cannot be reduced to hyperjerk dynamics, or they possess a functionally complicated hyperjerk form.

2 Transformation of 4-D systems to hyperjerk form

Consider a 4-D system of the form

$$\begin{cases} x' = c_1 + b_{11}x + b_{12}y + b_{13}z + b_{14}u + n_1\left(x\right) \\ y' = c_2 + b_{21}x + b_{22}y + b_{23}z + b_{24}u + n_2\left(x, y, z, u\right) \\ z' = c_3 + b_{31}x + b_{32}y + b_{33}z + b_{34}u + n_3\left(x, y, z, u\right) \\ u' = c_4 + b_{41}x + b_{42}y + b_{43}z + b_{44}u + n_4\left(x, y, z, u\right) \end{cases} \tag{2.1}$$

where the functions $n_1\left(x\right)$ and $\left(n_i\left(x, y, z, u\right)\right)_{2 \leq i \leq 4}$ are assumed to be at least of class C^2 in their corresponding arguments. We set $n_{11}\left(x\right) = \frac{\partial n_i(x)}{\partial x}, n_{111} = \frac{\partial^2 n_1(x)}{\partial x^2}, n_{i1}\left(x, y, z, u\right) = \frac{\partial n_i(x,y,z,u)}{\partial x}, n_{i2}\left(x, y, z, u\right) = \frac{\partial n_i(x,y,z,u)}{\partial y}, n_{i3}\left(x, y, z, u\right) = \frac{\partial n_i(x,y,z,u)}{\partial z}$, and $n_{i4}\left(x, y, z, u\right) = \frac{\partial n_i(x,y,z,u)}{\partial u}, i = 2, 3, 4$. In this paper, we derive only the hyperjerk form of system (1) with respect to x. The same logic applies for y, z, and u. For this purpose, assume that $b_{12} \neq 0$. Then from the first equation of (1), we have $y\left(x, x', z, u\right) = -\frac{1}{b_{12}}\left(-x' + c_1 + n_1\left(x\right) + b_{11}x + b_{13}z + b_{14}u\right)$. Thus $y'\left(x, x', x'', z, u\right) = f_1z + f_2x + f_3u + f_4 + f_5$, where

$$\left\{\begin{array}{c} f_1 = \frac{b_{13}^2 b_{32} - b_{12} b_{13} b_{33} - b_{12} b_{14} b_{43} + b_{13} b_{14} b_{42}}{b_{12}^2} \\ f_2 = \frac{b_{11} b_{13} b_{32} - b_{12} b_{13} b_{31} + b_{11} b_{14} b_{42} - b_{12} b_{14} b_{41}}{b_{12}^2} \\ f_3 = \frac{b_{14}^2 b_{42} - b_{12} b_{13} b_{34} + b_{13} b_{14} b_{32} - b_{12} b_{14} b_{44}}{b_{12}^2} \\ f_4\left(x, x', z, u\right) = -\frac{\left(b_{13}\left(c_3 + n_3((x,y,z,u)) - \frac{b_{32}\left(c_1 + n_1(x)\right)}{b_{12}}\right) + b_{14}\left(c_4 + n_4(x,y,z,u) - \frac{b_{42}\left(c_1 + n_1(x)\right)}{b_{12}}\right)\right)}{b_{12}} \\ f_5\left(x, x', x''\right) = -\frac{b_{11} b_{12} + b_{13} b_{32} + b_{14} b_{42} + b_{12} n_{11}(x)}{b_{12}^2} x' + \frac{1}{b_{12}} x'' \end{array}\right. \tag{2.2}$$

By equating the formula for $y' = y'\left(x, x', x'', z, u\right)$ with the second equation of (2.1), we obtain

$$\left(f_1 - b_{23} + \frac{1}{b_{12}} b_{13} b_{22}\right) z = n_2 - f_4 - f_6 \tag{2.3}$$

where

$$f_6\left(x, x', x''\right) = \frac{f_2 b_{12} + b_{11} b_{22} - b_{12} b_{21}}{b_{12}} x - \frac{b_{22}}{b_{12}} x' - c_2 + \frac{b_{22} c_1}{b_{12}} + \frac{b_{22}}{b_{12}} n_1(x) + f_5 \tag{2.4}$$

We remark that the second member of (2.3) depends on z in the part $n_2 - f_4$, so if we assume that $n_2(x, y, z, u) - f_4\left(x, x', z, u\right) = g_1(x, u)$, i.e., this part does not depend on the variable z, with g_1 being an arbitrary function of the indicated arguments, then

$$z\left(x, x', x''\right) = f_7 f_6\left(x, x', x''\right) \tag{2.5}$$

where f_7 is given by

$$f_7 = \frac{b_{12}^2}{b_{12}^2 b_{23} - b_{13}^2 b_{32} - b_{12} b_{13} b_{22} + b_{12} b_{13} b_{33} + b_{12} b_{14} b_{43} - b_{13} b_{14} b_{42}} \tag{2.6}$$

with the condition

$$b_{12}^2 b_{23} - b_{13}^2 b_{32} - b_{12} b_{13} b_{22} + b_{12} b_{13} b_{33} + b_{12} b_{14} b_{43} - b_{13} b_{14} b_{42} \neq 0 \tag{2.7}$$

The function g_1 does not depend on y since y itself depends on z. Hence the condition $n_2 - f_4 = g_1(x, u)$ is equivalent to

$$\left(b_{12} n_2 + b_{13} n_3 + b_{14} n_4\right)(x, y, z, u) = b_{12} g_1(x, u) - b_{12} f_8(x) \tag{2.8}$$

where

$$f_8(x) = \frac{\left(b_{13}\left(c_3 - \frac{b_{32}\left(c_1 + n_1(x)\right)}{b_{12}}\right) + b_{14}\left(c_4 - \frac{b_{42}\left(c_1 + n_1(x)\right)}{b_{12}}\right)\right)}{b_{12}} \tag{2.9}$$

From (2.5), we have $z' = f_7\left(\frac{1}{b_{12}} x''' + f_{10}(x) x'' + f_9\left(x, x'\right) x'\right)$, where

$$\left\{\begin{array}{c} f_9\left(x, x'\right) = \left(\frac{b_{12}\left(f_2 - b_{21}\right) + b_{22} b_{11} + b_{22} n_{11}(x) - n_{111}(x) x'}{b_{12}}\right) \\ f_{10}(x) = \left(-\frac{b_{22}}{b_{12}} - \frac{\left(b_{11} b_{12} + b_{13} b_{32} + b_{14} b_{42} + b_{12} n_{11}(x)\right)}{b_{12}^2}\right) \end{array}\right. \tag{2.10}$$

By equating the formula for z' with the third equation of (2.1), we obtain

$$-\frac{b_{12} b_{34} - b_{14} b_{32}}{b_{12}} u = -f_{11} - f_{12} - \frac{f_7}{b_{12}} x''' \tag{2.11}$$

where

$$\left\{\begin{array}{c} f_{11}\left(x, x', x''\right) = \frac{b_{11} b_{32} - b_{12} b_{31}}{b_{12}} x + \frac{-b_{32} + f_7 f_9\left(x, x'\right) b_{12}}{b_{12}} x' + f_7 f_{10}(x) x'' \\ f_{12}\left(x, x', x'', u\right) = -n_3(x, y, z, u) - c_3 + \frac{b_{32}\left(c_1 + n_1(x) + f_6 f_7 b_{13}\right)}{b_{12}} - f_6 f_7 b_{33} \end{array}\right. \tag{2.12}$$

We remark that the second member of (2.11) depends on u in the part $-f_{12}$, so if we assume that $-f_{12} = g_2(x, z)$, i.e., this part does not depend on the variable u, with g_2 being an arbitrary function of the indicated arguments, then

$$u = \frac{f_7 x''' + b_{12} f_{11}\left(x, x', x''\right) + b_{12} f_{12}\left(x, x', x''\right)}{b_{12} b_{34} - b_{14} b_{32}} \tag{2.13}$$

with the condition

$$b_{12}b_{34} - b_{14}b_{32} \neq 0 \tag{2.14}$$

Hence the condition $-f_{12} = g_2(x,z)$ is equivalent to

$$-n_3(x,y,z,u) - c_3 + \frac{b_{32}(c_1 + n_1(x) + f_6 f_7 b_{13})}{b_{12}} - f_6 f_7 b_{33} = g_2(x,z) \tag{2.15}$$

From (2.13), we have $u' = f_{13}x^{(4)} + f_{14}$, where

$$\begin{cases} f_{13} = \frac{f_7}{b_{12}b_{34}-b_{14}b_{32}} \\ f_{14}(x,x',x'',x''') = \frac{b_{12}f'_{11}(x,x',x'',x''')}{b_{12}b_{34}-b_{14}b_{32}} + \frac{b_{12}f'_{12}(x,x',x'',x''')}{b_{12}b_{34}-b_{14}b_{32}} \end{cases} \tag{2.16}$$

because $\frac{df_{11}(x,x',x'')}{dt} = f'_{11}(x,x',x'',x''')$, and since $f_{12} = -g_2(x,z)$, then $\frac{f_{12}(x,y,z,u)}{dt} = f'_{12}(x,x',x'',x''')$. Again, by equating the formula for u' with the fourth equation of (2.1), we obtain

$$x^{(4)} = \frac{f_{15}(x,x',x'',x''') + f_{16}(x,x',x'',x''')}{f_{13}} = H(x,x',x'',x''') \tag{2.17}$$

where

$$\begin{cases} f_{15}(x,x',x'',x''',u) = c_4 + b_{41}x + b_{43}f_7 f_6 + \rho_0 \\ f_{16}(x,x',x'',x''',u) = b_{42}\left(-\frac{1}{b_{12}}(-x' + c_1 + n_1(x) + b_{11}x + \rho_1)\right) \\ \rho_0(x,x',x'',x''',u) = b_{44}\left(\frac{f_7 x''' + b_{12}f_{11} + b_{12}f_{12}}{b_{12}b_{34}-b_{14}b_{32}}\right) + n_4 - f_{14} \\ \rho_1(x,x',x'',x''',u) = b_{13}f_7 f_6 + b_{14}\left(\frac{f_7 x''' + b_{12}f_{11} + b_{12}f_{12}}{b_{12}b_{34}-b_{14}b_{32}}\right) \end{cases} \tag{2.18}$$

Finally, the form (2.17) is the corresponding hyperjerk form of the 4-D dynamical system (2.1). Several chaotic examples of hyperjerk motion were studied in the literature where some of them are listed in [8] with many illustrations. For example, all periodically forced oscillators and some of the coupled oscillators are equivalent to a snap form as shown in [4]. This includes the frictionless forced pendulum and the periodically forced undamped oscillator with a cubic restoring force. Also, the same type of system with damping was studied in [8]. The simplest examples of chaotic hyperjerk systems have been studied in [1]. These include snap systems with one nonlinear function and the simplest dissipative chaotic case with a single quadratic nonlinearity given by $x'''' + 6x'' = 1 - x^2$.

3 The expression of the transformation

In this section, we derive a rigorous expression for the transformation between the 4-D dynamical system (2.1) and the hyperjerk form (2.17). Indeed, the above procedure defines an invertible transformation $T = T(x,x',x'',x''')$ provided the following conditions hold:

$$\begin{cases} b_{12} \neq 0 \\ b_{12}^2 b_{23} - b_{13}^2 b_{32} - b_{12}b_{13}b_{22} + b_{12}b_{13}b_{33} + b_{12}b_{14}b_{43} - b_{13}b_{14}b_{42} \neq 0 \\ b_{12}b_{34} - b_{14}b_{32} \neq 0 \\ n_2 - f_4 = g_1(x,u) \\ -n_3 - c_3 + \frac{1}{b_{12}}b_{32}(c_1 + n_1(x) + f_6 f_7 b_{13}) - f_6 f_7 b_{33} = g_2(x,z) \end{cases} \tag{3.1}$$

where g_1 and g_2 are arbitrary functions of the indicated arguments. The transformation $T(x,x',x'',x''')$ is defined by

$$\begin{cases} T_1(x,y,z,u) = x \\ T_2(x,y,z,u) = x' = c_1 + b_{11}x + b_{12}y + n_1(x) \\ T_3(x,y,z,u) = x'' = f_{17}(x,y) + f_{18}(x,z,u) + f_{19}(x,y,z,u) \\ T_4(x,y,z,u) = x''' = \Psi_1 + \Psi_2 \\ \Psi_1 = f_{20}x^3 + f_{21}x^2 y + f_{22}x^2 + f_{23}xy + f_{24}xz + f_{25}xu \\ \Psi_2 = f_{26}x + f_{27}y^2 + f_{28}yz + f_{29}yu + f_{30}y + f_{31}z + f_{32}u + f_{33} \end{cases} \tag{3.2}$$

and its inverse is defined by

$$\begin{cases} T_1^{-1}\left(x,x',x'',x'''\right)=x \\ T_4^{-1}\left(x,x',x'',x'''\right)=u=\frac{f_7x'''+b_{12}f_{11}\left(x,x',x''\right)+b_{12}f_{12}\left(x,x',x''\right)}{b_{12}b_{34}-b_{14}b_{32}} \\ T_3^{-1}\left(x,x',x'',x'''\right)=z=f_7f_6\left(x,x',x''\right) \\ T_2^{-1}\left(x,x',x'',x'''\right)=y=-\frac{1}{b_{12}}\left(-x'+c_1+n_1\left(x\right)+b_{11}x+b_{13}z+b_{14}u\right) \end{cases} \tag{3.3}$$

where

$$\begin{cases} f_{17}=\left(b_{11}^2+n_{11}\left(x\right)b_{11}+b_{12}b_{21}\right)x+\left(b_{11}b_{12}+b_{12}b_{22}+b_{12}n_{11}\left(x\right)\right)y \\ f_{18}=\left(b_{11}b_{13}+b_{12}b_{23}+b_{13}n_{11}\left(x\right)\right)z+\left(b_{11}b_{14}+b_{12}b_{24}+b_{14}n_{11}\left(x\right)\right)u \\ f_{19}=\left(b_{11}\left(c_1+n_1\left(x\right)\right)+b_{12}\left(c_2+n_2\left(x,y,z,u\right)\right)+n_{11}\left(x\right)\left(c_1+n_1\left(x\right)\right)\right) \\ f_{20}=b_{11}^2n_{111}\left(x\right) \\ f_{21}=b_{11}b_{12}n_{111}\left(x\right) \\ f_{22}=\xi_1+\xi_2 \\ f_{23}=\xi_3+\xi_4 \\ f_{24}=b_{12}b_{23}n_{11}\left(x\right)+b_{13}b_{33}n_{11}\left(x\right)+b_{14}b_{43}n_{11}\left(x\right)+b_{11}b_{13}n_{111}\left(x\right) \end{cases} \tag{3.4}$$

and

$$\begin{cases} f_{25}=b_{12}b_{24}n_{11}+b_{13}b_{34}n_{11}+b_{14}b_{44}n_{11}+b_{11}b_{14}n_{111} \\ f_{26}=\xi_5+\xi_6+\xi_7+\xi_8+\xi_9 \\ f_{27}=b_{12}^2n_{111} \\ f_{28}=b_{12}b_{13}n_{111} \\ f_{29}=b_{12}b_{14}n_{111} \\ f_{30}=\xi_{10}+\xi_{11}+\xi_{12} \\ f_{31}=\xi_{13}+\xi_{14} \\ f_{32}=\xi_{15}+\xi_{16} \\ f_{33}=\xi_{17}+\xi_{18}+\xi_{19}+\xi_{20} \end{cases} \tag{3.5}$$

where

$$\begin{cases} \xi_1=b_{11}\left(n_{11}^2+b_{11}n_{11}+c_1n_{111}+n_1n_{111}\right)\left(x\right) \\ \xi_2=b_{11}n_{111}\left(x\right)\left(c_1+n_1\left(x\right)\right)+\left(b_{12}b_{21}+b_{13}b_{31}+b_{14}b_{41}\right)n_{11}\left(x\right) \\ \xi_3=b_{12}\left(n_{11}^2\left(x\right)+b_{11}n_{11}\left(x\right)+\left(c_1+n_1\left(x\right)\right)n_{111}\left(x\right)\right) \\ \xi_4=\left(b_{12}b_{22}+b_{13}b_{32}+b_{14}b_{42}\right)n_{11}\left(x\right)+b_{11}b_{12}n_{111}\left(x\right) \\ \xi_5=\left(c_1+n_1\left(x\right)\right)\left(n_{11}^2\left(x\right)+b_{11}n_{11}\left(x\right)+\left(c_1+n_1\left(x\right)\right)n_{111}\left(x\right)\right) \\ \xi_6=b_{11}\left(b_{11}^2+\left(b_{11}+c_1+n_1\right)n_{11}\left(x\right)+b_{12}b_{21}+b_{12}n_{21}\left(x,y,z,u\right)\right) \\ \xi_7=b_{21}\left(b_{11}b_{12}+b_{12}b_{22}+b_{12}n_{22}\right)+b_{31}\left(b_{11}b_{13}+b_{12}b_{23}+b_{12}n_{23}\right) \\ \xi_8=b_{41}\left(b_{11}b_{14}+b_{12}b_{24}+b_{12}n_{24}\right)+b_{12}n_{11}\left(x\right)\left(c_2+n_2\right) \\ \xi_9=b_{13}n_{11}\left(x\right)\left(c_3+n_3\left(x,y,z,u\right)\right)+b_{14}n_{11}\left(c_4+n_4\left(x,y,z,u\right)\right) \\ \xi_{10}=b_{12}\left(b_{11}^2+n_{11}b_{11}+c_1n_{11}+n_1n_{11}+b_{12}b_{21}+b_{12}n_{21}\right) \end{cases} \tag{3.6}$$

and

$$\begin{cases} \xi_{11}=b_{22}\left(b_{11}b_{12}+b_{12}b_{22}+b_{12}n_{22}\right)+b_{32}\left(b_{11}b_{13}+b_{12}b_{23}+b_{12}n_{23}\right) \\ \xi_{12}=b_{42}\left(b_{11}b_{14}+b_{12}b_{24}+b_{12}n_{24}\right)+b_{12}n_{111}\left(c_1+n_1\right) \\ \xi_{13}=b_{23}\left(b_{11}b_{12}+b_{12}b_{22}+b_{12}n_{22}\right)+b_{33}\left(b_{11}b_{13}+b_{12}b_{23}+b_{12}n_{23}\right) \\ \xi_{14}=b_{43}\left(b_{11}b_{14}+b_{12}b_{24}+b_{12}n_{24}\right)+b_{13}n_{111}\left(c_1+n_1\right) \\ \xi_{15}=b_{24}\left(b_{11}b_{12}+b_{12}b_{22}+b_{12}n_{22}\right)+b_{34}\left(b_{11}b_{13}+b_{12}b_{23}+b_{12}n_{23}\right) \\ \xi_{16}=b_{44}\left(b_{11}b_{14}+b_{12}b_{24}+b_{12}n_{24}\right)+b_{14}n_{111}\left(c_1+n_1\right) \\ \xi_{17}=\left(c_1+n_1\right)\left(b_{11}^2+n_{11}b_{11}+c_1n_{11}+n_1n_{11}+b_{12}b_{21}+b_{12}n_{21}\right) \\ \xi_{18}=\left(c_2+n_2\left(x,y,z,u\right)\right)\left(b_{11}b_{12}+b_{12}b_{22}+b_{12}n_{22}\left(x,y,z,u\right)\right) \\ \xi_{19}=\left(c_4+n_4\left(x,y,z,u\right)\right)\left(b_{11}b_{14}+b_{12}b_{24}+b_{12}n_{24}\left(x,y,z,u\right)\right) \\ \xi_{20}=\left(c_3+n_3\left(x,y,z,u\right)\right)\left(b_{11}b_{13}+b_{12}b_{23}+b_{12}n_{23}\left(x,y,z,u\right)\right) \end{cases} \tag{3.7}$$

Note that the inverse transformation exits since the procedure described in Sec. 2 transforms the 4-D dynamical system (2.1) to the hyperjerk form in (2.17) without any singularities. Therefore, the inverse procedure defined by (3.3) can give the initial 4-D dynamical system (2.1).

4 Examples of hyperjerk dynamics

Using the above method, we can prove that any dynamical system of the functional form

$$\begin{cases} x' = c_1 + b_{11}x + b_{12}y + n_1(x) \\ y' = c_2 + b_{21}x + b_{22}y + b_{23}z + n_2(x,y) \\ z' = c_3 + b_{31}x + b_{32}y + b_{33}z + b_{34}u + n_3(x,y,z) \\ u' = c_4 + b_{41}x + b_{42}y + b_{43}z + b_{44}u + n_4(x,y,z,u) \end{cases} \tag{4.1}$$

can be reduced to a hyperjerk dynamic form $x'''' = H(x, x', x'', x''')$ if the following conditions hold:

$$b_{12} \neq 0, b_{23} \neq 0, b_{34} \neq 0 \tag{4.2}$$

That is, $b_{13} = b_{14} = b_{24} = 0$ in system (2.1). First, we notice that if n_2 and n_3 do not depend on z and u, respectively, then there is no need to consider the functions g_1 and g_2 defined above. In this case, we have $y(x, x') = -\frac{1}{b_{12}}(-x' + c_1 + n_1(x) + b_{11}x)$. Thus $y'(x, x', x'') = f_5 = -\frac{b_{11}b_{12}+b_{12}n_{11}(x)}{b_{12}^2}x' + \frac{1}{b_{12}}x''$, and $f_1 = 0, f_2 = 0, f_3 = 0, f_4 = 0$. Also, $z(x, x', x'') = f_7 f_6$, where $f_6 = \left(\frac{b_{11}b_{22}-b_{12}b_{21}}{b_{12}}\right)x - \frac{b_{22}}{b_{12}}x' - c_2 + \frac{b_{22}c_1}{b_{12}} + \frac{b_{22}}{b_{12}}n_1(x) + f_5, f_7 = \frac{1}{b_{23}}, f_8(x) = 0$. Hence $z' = f_7\left(\frac{1}{b_{12}}x''' + f_{10}(x)x'' + f_9 x'\right)$, where $f_9 = \left(\frac{b_{12}(f_2-b_{21})+b_{22}b_{11}+b_{22}n_{11}(x)-n_{111}(x)x'}{b_{12}}\right)$ and $f_{10}(x) = \left(-\frac{b_{22}}{b_{12}} - \frac{(b_{11}+n_{11}(x))}{b_{12}}\right)$. By equating the formula for z' with the third equation of (26), we obtain $u = \frac{f_7x'''+b_{12}f_{11}+b_{12}f_{12}}{b_{12}b_{34}}$, where $f_{11} = \frac{b_{11}b_{32}-b_{12}b_{31}}{b_{12}}x + \frac{-b_{32}+f_7f_9(x,x')b_{12}}{b_{12}}x' + f_7 f_{10}(x)x''$ and $f_{12} = -n_3 - c_3 + \frac{b_{32}(c_1+n_1(x))}{b_{12}} - f_6 f_7 b_{33}$. Hence we have $u' = f_{13}x^{(4)} + f_{14}$, where $f_{13} = \frac{f_7}{b_{12}b_{34}}$ and $f_{14} = \frac{f'_{11}}{b_{34}} + \frac{f'_{12}}{b_{34}}$. Finally, we obtain $x^{(4)} = \frac{f_{15}+f_{16}}{f_{13}} = H(x, x', x'', x''')$, where $f_{15} = c_4 + b_{41}x + b_{43}f_7 f_6 + \rho_0$ and $f_{16} = b_{42}\left(-\frac{1}{b_{12}}(-x' + c_1 + n_1(x) + b_{11}x)\right)$ with $\rho_0 = b_{44}\left(\frac{f_7x'''+b_{12}f_{11}+b_{12}f_{12}}{b_{12}b_{34}}\right) + n_4 - f_{14}$. In this case, the expression for the transformation $T = T(x, x', x'', x''')$ is given by

$$\begin{cases} T_1(x,y,z,u) = x \\ T_2(x,y,z,u) = x' = c_1 + b_{11}x + b_{12}y + n_1(x) \\ T_3(x,y,z,u) = x'' = f_{17}(x,y) + f_{18}(x,z,u) + f_{19}(x,y,z,u) \\ T_4(x,y,z,u) = x''' = \Psi_1 + \Psi_2 \\ \Psi_1 = f_{20}x^3 + f_{21}x^2y + f_{22}x^2 + f_{23}xy + f_{24}xz \\ \Psi_2 = f_{26}x + f_{27}y^2 + f_{30}y + f_{31}z + f_{32}u + f_{33} \end{cases} \tag{4.3}$$

and its inverse is defined by

$$\begin{cases} T_1^{-1}(x,x',x'',x''') = x \\ T_4^{-1}(x,x',x'',x''') = u = \frac{f_7x'''+b_{12}f_{11}(x,x',x'')+b_{12}f_{12}(x,x',x'')}{b_{12}b_{34}} \\ T_3^{-1}(x,x',x'',x''') = z = f_7 f_6(x,x',x'') \\ T_2^{-1}(x,x',x'',x''') = y = -\frac{1}{b_{12}}(-x' + c_1 + n_1(x) + b_{11}x) \end{cases} \tag{4.4}$$

where

$$\begin{cases} f_{17} = \left(b_{11}^2 + n_{11}(x)b_{11} + b_{12}b_{21}\right)x + \left(b_{11}b_{12} + b_{12}b_{22} + b_{12}n_{11}(x)\right)y \\ f_{18} = b_{12}b_{23}z \\ f_{19} = \left(b_{11}(c_1 + n_1(x)) + b_{12}(c_2 + n_2(x,y)) + n_{11}(x)(c_1 + n_1(x))\right) \\ f_{20} = b_{11}^2 n_{111}(x), f_{21} = b_{11}b_{12}n_{111}(x), f_{22} = \xi_1 + \xi_2 \\ f_{23} = \xi_3 + \xi_4, f_{24} = b_{12}b_{23}n_{11}(x), f_{25} = 0, f_{28} = 0, f_{29} = 0 \\ f_{26} = \xi_5 + \xi_6 + \xi_7 + \xi_8, f_{27} = b_{12}^2 n_{111} \\ f_{30} = \xi_{10} + \xi_{11} + \xi_{12}, f_{31} = \xi_{13} + \xi_{14}, f_{32} = \xi_{15} + \xi_{16} \\ f_{33} = \xi_{17} + \xi_{18} + \xi_{19} + \xi_{20} \end{cases} \tag{4.5}$$

where

$$\begin{cases} \xi_1 = b_{11}\left(n_{11}^2 + b_{11}n_{11} + c_1 n_{111} + n_1 n_{111}\right)(x) \\ \xi_2 = b_{11}n_{111}(x)(c_1 + n_1(x)) + (b_{12}b_{21})\, n_{11}(x) \\ \xi_3 = b_{12}\left(n_{11}^2(x) + b_{11}n_{11}(x) + (c_1 + n_1(x))\, n_{111}(x)\right) \\ \xi_4 = (b_{12}b_{22})\, n_{11}(x) + b_{11}b_{12}n_{111}(x) \\ \xi_5 = (c_1 + n_1(x))\left(n_{11}^2(x) + b_{11}n_{11}(x) + (c_1 + n_1(x))\, n_{111}(x)\right) \\ \xi_6 = b_{11}\left(b_{11}^2 + (b_{11} + c_1 + n_1)\, n_{11}(x) + b_{12}b_{21} + b_{12}n_{21}(x,y)\right) \\ \xi_7 = b_{21}(b_{11}b_{12} + b_{12}b_{22} + b_{12}n_{22}(x,y)) + b_{31}(b_{12}b_{23} + b_{12}n_{23}(x,y)) \\ \xi_8 = b_{41}(b_{12}n_{24}(x,y)) + b_{12}n_{11}(x)(c_2 + n_2(x,y)) \\ \xi_9 = 0 \\ \xi_{10} = b_{12}\left(b_{11}^2 + n_{11}b_{11} + c_1 n_{11} + n_1 n_{11} + b_{12}b_{21} + b_{12}n_{21}\right) \end{cases} \tag{4.6}$$

and

$$\begin{cases} \xi_{11} = b_{22}(b_{11}b_{12} + b_{12}b_{22} + b_{12}n_{22}) + b_{32}(b_{12}b_{23} + b_{12}n_{23}) \\ \xi_{12} = b_{42}(b_{12}n_{24}) + b_{12}n_{111}(c_1 + n_1) \\ \xi_{13} = b_{23}(b_{11}b_{12} + b_{12}b_{22} + b_{12}n_{22}) + b_{33}(b_{12}b_{23} + b_{12}n_{23}) \\ \xi_{14} = b_{43}(b_{12}n_{24}) \\ \xi_{15} = b_{34}(b_{12}b_{23} + b_{12}n_{23}) \\ \xi_{16} = b_{44}b_{12}n_{24} \\ \xi_{17} = (c_1 + n_1)\left(b_{11}^2 + n_{11}b_{11} + c_1 n_{11} + n_1 n_{11} + b_{12}b_{21} + b_{12}n_{21}\right) \\ \xi_{18} = (c_2 + n_2(x,y))(b_{11}b_{12} + b_{12}b_{22} + b_{12}n_{22}(x,y)) \\ \xi_{19} = (c_4 + n_4(x,y,z,u))(b_{12}n_{24}(x,y)) \\ \xi_{20} = (c_3 + n_3(x,y,z))(b_{12}b_{23} + b_{12}n_{23}(x,y)) \end{cases} \tag{4.7}$$

We remark that if $n_1(x) = 0$, then the expression for the transformation $T = T(x, x', x'', x''')$ is given by

$$\begin{cases} T_1(x,y,z,u) = x \\ T_2(x,y,z,u) = x' = c_1 + b_{11}x + b_{12}y \\ T_3(x,y,z,u) = x'' = f_{17}(x,y) + f_{18}(z) + f_{19}(x,y) \\ T_4(x,y,z,u) = x''' = f_{26}x + f_{30}y + f_{31}z + f_{32}u + f_{33}(x,y,z,u) \end{cases} \tag{4.8}$$

and its inverse is defined by

$$\begin{cases} T_1^{-1}(x,x',x'',x''') = x \\ T_4^{-1}(x,x',x'',x''') = u = \frac{f_7 x''' + b_{12}f_{11}\left(x,x',x''\right) + b_{12}f_{12}\left(x,x',x''\right)}{b_{12}b_{34}} \\ T_3^{-1}(x,x',x'',x''') = z = f_7 f_6(x,x',x'') \\ T_2^{-1}(x,x',x'',x''') = y = -\frac{1}{b_{12}}(-x' + c_1 + n_1(x) + b_{11}x) \end{cases} \tag{4.9}$$

where

$$\begin{cases} f_{17} = \left(b_{11}^2 + b_{12}b_{21}\right)x + (b_{11}b_{12} + b_{12}b_{22})\, y \\ f_{18} = b_{12}b_{23}z,\ f_{19} = (b_{11}(c_1) + b_{12}(c_2 + n_2(x,y))) \\ f_{20} = 0, f_{21} = 0, f_{22} = 0, f_{23} = 0, f_{24} = 0, f_{25} = 0, f_{28} = 0, f_{29} = 0, f_{27} = 0 \\ f_{26} = \xi_6 + \xi_7 + \xi_8,\ f_{30} = \xi_{10} + \xi_{11} + \xi_{12},\ f_{31} = \xi_{13} + \xi_{14} \\ f_{32} = \xi_{15} + \xi_{16},\ f_{33} = \xi_{17} + \xi_{18} + \xi_{19} + \xi_{20} \end{cases} \tag{4.10}$$

where

$$\begin{cases} \xi_1 = 0, \xi_2 = 0, \xi_3 = 0, \xi_4 = 0, \xi_5 = 0, \xi_9 = 0 \\ \xi_6 = b_{11}\left(b_{11}^2 + b_{12}b_{21} + b_{12}n_{21}(x,y)\right) \\ \xi_7 = b_{21}(b_{11}b_{12} + b_{12}b_{22} + b_{12}n_{22}(x,y)) + b_{31}(b_{12}b_{23} + b_{12}n_{23}(x,y)) \\ \xi_8 = b_{41}(b_{12}n_{24}(x,y)),\ \xi_{10} = b_{12}\left(b_{11}^2 + b_{12}b_{21} + b_{12}n_{21}(x,y)\right) \\ \xi_{11} = b_{22}(b_{11}b_{12} + b_{12}b_{22} + b_{12}n_{22}(x,y)) + b_{32}(b_{12}b_{23} + b_{12}n_{23}(x,y)) \\ \xi_{12} = b_{42}(b_{12}n_{24}) \\ \xi_{13} = b_{23}(b_{11}b_{12} + b_{12}b_{22} + b_{12}n_{22}(x,y)) + b_{33}(b_{12}b_{23} + b_{12}n_{23}(x,y)) \\ \xi_{14} = b_{43}(b_{12}n_{24}) \end{cases} \tag{4.11}$$

and

$$\begin{cases} \xi_{15} = b_{34}\left(b_{12}b_{23} + b_{12}n_{23}\left(x, y\right)\right) \\ \xi_{16} = b_{44}b_{12}n_{24}\left(x, y\right) \\ \xi_{17} = c_1\left(b_{11}^2 + b_{12}b_{21} + b_{12}n_{21}\left(x, y\right)\right) \\ \xi_{18} = \left(c_2 + n_2\left(x, y\right)\right)\left(b_{11}b_{12} + b_{12}b_{22} + b_{12}n_{22}\left(x, y\right)\right) \\ \xi_{19} = b_{12}n_{24}\left(x, y\right)\left(c_4 + n_4\left(x, y, z, u\right)\right) \\ \xi_{20} = \left(c_3 + n_3\left(x, y, z\right)\right)\left(b_{12}b_{23} + b_{12}n_{23}\left(x, y\right)\right) \end{cases} \tag{4.12}$$

Now we find sufficient conditions for the equivalence of a 4-D quadratic system with a hyperjerk system. According to the previous analysis, the general 4-D quadratic system has the following form

$$\begin{cases} x' = c_1 + b_{11}x + b_{12}y + n_1\left(x\right) \\ y' = c_2 + b_{21}x + b_{22}y + b_{23}z + n_2\left(x, y, z, u\right) \\ z' = c_3 + b_{31}x + b_{32}y + b_{33}z + b_{34}u + n_3\left(x, y, z, u\right) \\ u' = c_4 + b_{41}x + b_{42}y + b_{43}z + b_{44}u + n_4\left(x, y, z, u\right) \end{cases} \tag{4.13}$$

where

$$\begin{cases} n_1\left(x\right) = a_4x^2 \\ n_2 = b_4x^2 + b_5y^2 + b_6z^2 + b_7u^2 + b_8xy + b_9xz + b_{10}yz + b_{11}xu + b_{12}zu + b_{13}yu \\ n_3 = e_4x^2 + e_5y^2 + e_6z^2 + e_7u^2 + e_8xy + e_9xz + e_{10}yz + e_{11}xu + e_{12}zu + e_{13}yu \\ n_4 = d_4x^2 + d_5y^2 + d_6z^2 + d_7u^2 + d_8xy + d_9xz + d_{10}yz + d_{11}xu + d_{12}zu + d_{13}yu \end{cases} \tag{4.14}$$

By using the form (4.1), it is easy to show that any 4-D quadratic system of the form

$$\begin{cases} x' = c_1 + b_{11}x + b_{12}y + a_4x^2 \\ y' = c_2 + b_{21}x + b_{22}y + b_{23}z + b_4x^2 + b_5y^2 + b_8xy \\ z' = c_3 + b_{31}x + b_{32}y + b_{33}z + b_{34}u + e_4x^2 + e_5y^2 + e_6z^2 + e_8xy + e_9xz + e_{10}yz \\ u' = c_4 + b_{41}x + b_{42}y + b_{43}z + b_{44}u + n_4\left(x, y, z, u\right) \end{cases} \tag{4.15}$$

can be reduced to a hyperjerk dynamic form $x'''' = H\left(x, x', x'', x'''\right)$ if the conditions (4.2) hold. We remark that the form (4.15) contains 20 nonlinearities, and the best known hyperchaotic 4-D quadratic system is the Rössler system that contains one nonlinearity xz and is given by $x' = -y-z, y' = x+ay+u, z' = b+xz, u' = cu-dz$, but this system is not equivalent to any hyperjerk system due to the presence of singularities. In [8] the algebraically simplest hyperchaotic snap system was studied, and it is given by $x'''' + x^4x''' + Ax'' + x' + x = 0$. This system displays hyperchaos when $A = 3.6$ with Lyapunov exponents $(0.1310, 0.0358, 0, -1.2550)$.

5 Conclusion

The possible cases for equivalence between four-dimensional autonomous dynamical systems and hyperjerk dynamics were studied in this paper. Some examples were also presented and discussed.

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Author information

Zeraoulia Elhadj, Department of Mathematics, University of Tébessa, (12002), Algeria.
E-mail: zeraoulia@mail.univ-tebessa.dz

J. C. Sprott, Department of Physics, University of Wisconsin, Madison, WI 53706, U.S.A..
E-mail: sprott@physics.wisc.edu

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