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Spline-based numerical treatments of Bratu-type equations

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1 Abstract

Three different spline-based approaches for solving Bratu and Bratu-type equations are presented. The classical cubic spline collocation method, an adaptive spline collocation on nonuniform partitions, and an optimal collocation method are derived for solving Bratu-type equations. Numerical examples are presented to verify the efficiency and accuracy of these methods when compared to other numerical schemes. The fourth order of convergence for the optimal method is verified.

2 Introduction

A nonlinear elliptic eigenvalue problem has the form

$$\begin{aligned} &\Delta u(x) + \lambda f(u(x)) = 0, \quad x \in \Omega \\ &u(x) = 0, \quad x \in \partial\Omega \end{aligned} \tag{2.1}$$

Equation (2.1) arises in many fields of science and engineering such as radiative heat transfer, combustion theory, and nanotechnology [4, 12, 13, 9].

Bratu equation, which is a special case of equation (2.1), is a boundary value problem in one-dimensional planar coordinates that has the form

$$\begin{aligned} &u'' + \lambda e^u = 0, \quad x \in [0,1] \\ &u(0) = u(1) = 0 \end{aligned} . \tag{2.2}$$

For $\lambda > 0$, the exact solution of equation (2.2) is given by [3]

$$u(x) = -2\ln\left[\frac{\cosh\left(\frac{\theta}{2}\left(x-\frac{1}{2}\right)\right)}{\cosh\left(\frac{\theta}{4}\right)}\right] \tag{2.3}$$

where θ satisfies

$$\theta = \sqrt{2\lambda}\cosh\left(\frac{\theta}{4}\right). \tag{2.4}$$

Bratu equation posses two solutions, one solution, or no solution provided that $\lambda < \lambda_c,\ \lambda = \lambda_c,$ or $\lambda > \lambda_c$, respectively, where $\lambda_c,$ called the critical value, satisfies

$$1 = \frac{1}{4}\sqrt{2\lambda_c}\sinh\left(\frac{\theta}{4}\right). \tag{2.5}$$

The numerical value of λ_c, which can be confirmed by any computer algebra software is given by $\lambda_c = 3.513830719$ [4].

In addition to Bratu equation (2.2), we will examine the Bratu-type boundary value problem

$$\begin{aligned} &u'' - \pi^2 e^u = 0, \quad 0 < x < 1 \\ &u(0) = u(1) = 0 \end{aligned} , \tag{2.6}$$

and the Bratu-type initial value problem

$$\begin{aligned} &u'' - 2e^u = 0, \quad 0 < x < 1 \\ &u(0) = u'(0) = 0 \end{aligned} . \tag{2.7}$$

Approximate solutions of Bratu and Bratu-type equations have been investigated by several authors (see for example [1, 3, 4, 10, 11, 12, 13]). In a recent study, Aksoy et al [2] developed a perturbation solution to Bratu-type equations. In the current study, cubic B-spline collocations are employed over uniform and nonuniform meshes to solve Bratu and Bratu-type equations. Using uniform meshes produce a system of nonlinear equations, which is usually difficult to solve, even with a limited number of mesh points. The level of difficulty increases substantially if the number of mesh points is increased as to improve accuracy. In this paper, an appropriate graded mesh will be defined on nonuniform meshes in order to produce sufficiently accurate results in an efficient way. Notice that this technique was mainly used to solve boundary value problems with regions of large variations by placing more points of partitions in these regions, and hence permit a wider choice for the points of interpolation [8], and hence leads to acceptable approximate solutions near layer points.

The adaptive spline collocation on nonuniform meshes is restricted to linear boundary value problems. Thus, the implementation of this approach on the nonlinear Bratu-type equations requires a preliminary step, which is to begin by applying an iteration scheme that arise from the implementation of Newton's method.

In section 4, the guidelines presented in [6] were closely followed to derive an optimal spline collocation method to find approximate solution for equations of the form (2.6)–(2.7). The convergence and stability analysis of these methods can be found in some references, see for example [6, 7, 8].

3 Cubic B-spline collocation method

Let π be a uniform partition of the interval $[0, 1]$ given by

$$\pi : 0 = x_0 < x_1 < ... < x_n < x_{n+1} = 1.$$

That is, $x_i = ih, i = 0, 1, 2, ..., n + 1$, and $h = \frac{1}{n+1}$. By including an additional point at each side of the partition π, the cubic B-spline basis functions $\{\widehat{B}_i(x)\}_{i=-1}^{n+2}$ are defined as follows

$$\widehat{B}_i(x) = \frac{1}{h^3}\begin{cases} (x-x_{i-2})^3, & [x_{i-2}, x_{i-1}] \\ h^3 + 3h^2(x-x_{i-1}) + 3h(x-x_{i-1})^2 \\ -3(x-x_{i-1})^3, & [x_{i-1}, x_i] \\ h^3 + 3h^2(x_{i+1}-x) + 3h(x_{i+1}-x)^2 \\ -3(x_{i+1}-x)^3, & [x_i, x_{i+1}] \\ (x_{i+2}-x)^3, & [x_{i+1}, x_{i+2}] \\ 0, & \text{otherwise} \end{cases} \tag{3.1}$$

where $i = -1, 0, \cdots, n + 2$. Each basis function $\widehat{B}_i(x) \in C^2[a, b]$ and the values of of $\widehat{B}_i(x), \widehat{B}'_i(x)$ and $\widehat{B}''_i(x)$ at the nodal points $x_i = ih$ are given in Table 1.

Table 1
Cubic B-spline basis values

	$\widehat{B}_i$	$\widehat{B}'_i$	$\widehat{B}''_i$
x_{i-2}	0	0	0
x_{i-1}	1	$3/h$	$6/h^2$
x_i	4	0	$-12/h^2$
x_{i+1}	1	$-3/h$	$6/h^2$
x_{i+2}	0	0	0

To accommodate the boundary conditions in (2.2), we modify the splines as follows

$$\begin{aligned} B_0(x) &= \widehat{B}_0(x) - 4\widehat{B}_{-1}(x) \\ B_1(x) &= \widehat{B}_1(x) - \widehat{B}_{-1}(x) \\ B_i(x) &= \widehat{B}_i(x), \qquad i = 2, \cdots, n-1 \\ B_n(x) &= \widehat{B}_n(x) - \widehat{B}_{n+2}(x) \\ B_{n+1}(x) &= \widehat{B}_{n+1}(x) - 4\widehat{B}_{n+2}(x) \end{aligned} \tag{3.2}$$

Similarly, the initial conditions in (2.7) can be attained.
The approximate spline solution is expressed by

$$u(x) =_{i=0}^{n+1} a_i B_i(x) \tag{3.3}$$

Substituting the approximate solution (3.3) into equation (2.2) yields

$$ {}_{i=0}^{n+1} a_i B_i''(x_j) + \lambda \exp\left({}_{i=0}^{n+1} a_i B_i(x_j) \right) = 0, \quad j = 0, 1, \cdots, n+1 \tag{3.4} $$

Using (3.2), the $n+2$ by $n+2$ nonlinear system given by (3.4) is expressed in the matrix form

$$ \mathbf{AB} + \lambda\, \mathbf{F} = \mathbf{O} \tag{3.5} $$

where B is an $n+2$ by $n+2$ matrix, A is the $n+2$ vector with constant components a_i, F is the $n+2$ vector with components exp$(u(x_i))$, and G is the $n+2$ zero vector, that is

$$ \mathbf{B} = \frac{6}{h^2} \begin{bmatrix} -2 & 1 & 0 & 0 & 0 & ... & 0 \\ 1 & -2 & 1 & 0 & 0 & ... & 0 \\ 0 & 1 & -2 & 1 & 0 & ... & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & ... & 1 & -2 & 1 \\ 0 & 0 & 0 & ... & 0 & 1 & -2 \end{bmatrix} $$

$$ \mathbf{F} = \begin{bmatrix} \exp(4a_0 + a_1) \\ \exp(a_0 + 4a_1 + a_2) \\ \exp(a_1 + 4a_2 + a_3) \\ \vdots \\ \exp(a_{n-1} + 4a_n + a_{n+1}) \\ \exp(a_n + 4_{n+1}) \end{bmatrix}, \ \mathbf{A} = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n+1} \end{bmatrix}, \text{ and } \mathbf{O} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} $$

Notice that for large n, it is a difficult task to obtain an approximate solution to the nonlinear system (3.5).

4 Adaptive spline collocation

The method of adaptive spline collocation on nonuniform partitions has been developed by C. Christara et al in 2005 [6, 7] and was mainly applied to linear boundary value problems. The method is based on a function w that maps uniform partition points x_i to nonuniform ones w_i. It was proved that when the mapping function is such that more points are placed in regions of large variation of the solution to BVPs and fewer in other regions, the observed errors are much smaller than their counterparts when the same total number of equidistant points are used. It remains difficult, however, to identify an appropriate mapping function under realistic situations. Convergence and stability of these methods have been also established in [6, 7, 8].

The adaptive spline approach begins by choosing a strictly increasing bijective function, called grading function, $w(x) : [0, 1] \to [0, 1]$ that maps the uniform nodes $x_i \in [0, 1]$ into nonuniform nodes $w_i \in [0, 1]$. In an ideal situation, one would like these nonuniform nodes to be redistributed so as to obtain same error magnitude at each step (see [5]). For Bratu-type equations, define the grading function $w(x) : [0, 1] \to [0, 1]$ by

$$ w(x) = \frac{k - (1+k)^{1-x} + 1}{k}. \tag{4.1} $$

The grading function $w(x)$ redistributes the nodes with more points near $x = 1$, as k increases. To solve the boundary-value problem:

$$\begin{aligned} & r(w)u'' + p(w)u' + q(w)u = g(w) \\ & \alpha_0 u(0) + \beta_0 u'(0) = \gamma_0 \\ & \alpha_1 u(1) + \beta_1 u'(1) = \gamma_1 \end{aligned} \tag{4.2}$$

by the adaptive technique, we approximate $u(x)$ by

$$S(w) =_{i=-3}^{n-1} a_i \psi_i(w) \tag{4.3}$$

where $\psi_i(w)$ is the nonuniform spline function defined by

$$\psi_i(w) = \begin{cases} \frac{(x-w_i)^3}{(w_{i+3}-w_i)(w_{i+2}-w_i)(w_{i+1}-w_i)}, & w_i \le x \le w_{i+1} \\ \frac{x-w_i}{w_{i+3}-w_i}\left(\frac{(x-w_i)(w_{i+2}-x)}{(w_{i+2}-w_i)(w_{i+2}-w_{i+1})} + \frac{(x-w_{i+1})(w_{i+3}-x)}{(w_{i+3}-w_{i+1})(w_{i+2}-w_{i+1})}\right) + \\ \frac{(x-w_{i+1})^2(w_{i+4}-x)}{(w_{i+4}-w_{i+1})(w_{i+3}-w_{i+1})(w_{i+2}-w_{i+1})}, & w_{i+1} \le x \le w_{i+2} \\ \frac{w_{i+4}-x}{w_{i+4}-w_{i+1}}\left(\frac{(x-w_{i+1})(w_{i+3}-x)}{(w_{i+3}-w_{i+1})(w_{i+3}-w_{i+2})} + \frac{(x-w_{i+2})(w_{i+4}-x)}{(w_{i+4}-w_{i+2})(w_{i+3}-w_{i+2})}\right) + \\ \frac{(x-w_i)(w_{i+3}-x)^2}{(w_{i+3}-w_i)(w_{i+3}-w_{i+1})(w_{i+3}-w_{i+2})}, & w_{i+2} \le x \le w_{i+3} \\ \frac{(w_{i+4}-x)^3}{(w_{i+4}-w_{i+1})(w_{i+4}-w_{i+2})(w_{i+4}-w_{i+3})}, & w_{i+3} \le x \le w_{i+4} \\ 0, & \text{otherwise} \end{cases} \tag{4.4}$$

The following values of the spline functions together with their first and second derivatives at the nodes $w_{i+1}, w_{i+2}, ..., w_{i+3}$ are to be determined

$$S_i^{(r)} = \left[S_i^{(r)}(w_{i+1}),\ S_i^{(r)}(w_{i+2}),\ w_i^{(r)}(w_{i+3})\right], \tag{4.5}$$

also, the coefficients a_i in (4.3) are obtained by solving the linear system

$$\begin{aligned} & r(w_j)\left\{S''_{j-3,3}\, c_{j-3} + S''_{j-2,2}\, c_{j-2} + S''_{j-1,1}\, c_{j-1}\right\} + \\ & p(w_j)\left\{S'_{j-3,3}\, c_{j-3} + S'_{j-2,2}\, c_{j-2} + S'_{j-1,1}\, c_{j-1}\right\} + \\ & q(w_j)\left\{S_{j-3,3}\, c_{j-3} + S_{j-2,2}\, c_{j-2} + S_{j-1,1}\, c_{j-1}\right\} = g(w_j), \end{aligned} \tag{4.6}$$

for $j = 0, 1, 2, ..., n$.

For Bratu-type equations, the subsequent iteration scheme arising from Newton's method is firstly used, and then the resulting linear equation is solved by the adaptive technique

$$\begin{aligned} & u''_m - \lambda^2 \cosh(\lambda u_{m-1})\, u_m = \lambda \sinh(\lambda u_{m-1}) - \lambda^2 \cosh(\lambda u_{m-1})\, u_{m-1} \\ & u(0) = 0, \quad u(1) = 1 \end{aligned} \tag{4.7}$$

In other words, we start with $u_0 = u(0)$ and solve (4.6) where $r(w) = 1,\ p(w) = 0$, and

$$\begin{aligned} & q(w) = -\lambda^2 \cosh(\lambda u_{m-1}), \\ & g(w) = \lambda \sinh(\lambda u_{m-1}) - \lambda^2 \cosh(\lambda u_{m-1})\, u_{m-1}, \end{aligned} \tag{4.8}$$

for $m = 1, 2, 3, 4, ..., M$.

5 Optimal spline collocation method

Consider the BVP

$$Lu = ru'' + pu' + qu = g \text{ in } \Omega = (0,1) \tag{5.1}$$

$$Bu = \begin{cases} \alpha_0 u(0) + \beta_0 u'(0) = \gamma_0 \\ \alpha_1 u(1) + \beta_1 u'(1) = \gamma_1 \end{cases} \text{ on } \delta\Omega \tag{5.2}$$

where r, p, q, and g are functions of x, and α_i, β_i, and γ_i $(i = 1, 2)$ are scalars.

Let Δ be a partition of Ω and determine a spline u_Δ that satisfies (5.1)-(5.2). The one step method described in [6] is employed as follows:

$$\begin{aligned} Lu_\Delta + P_L u_\Delta &= g \text{ in } T_L \\ Bu_\Delta + P_B u_\Delta &= \gamma \text{ on } T_B \end{aligned} \tag{5.3}$$

where T are collocation points prescribed in $\overline{\Omega} = \Omega \cup \delta\Omega$, $T_B = T \cap \delta\Omega$, and $T_L = T - T_B$.

Let $h = \frac{1}{n}$, $x_i = ih,\ i = 0, 1, ..., n$. Let $w : [0, 1] \to [0, 1]$ be a grading function defined by $w(x_i) = w_i$ and define $h_i = w_{i+1} - w_i$, $i = 0, 1, ..., n$. Define

$$LS(w_i) = g(w_i) - P_L S(w_i),\ i = 0, 1, ..., n \tag{5.4}$$

where

$$\begin{aligned} P_L S(w_0) &= \frac{r(w_0)}{24} h_0(5h_0 - 4h_1 + h_2)\frac{(h_0 + h_1) \sqcap S''(w_1) - h_0 \sqcap S''(w_2)}{h_1}, \\ P_L S(w_i) &= \frac{r(w_i)}{12} h_i h_{i-1} \sqcap S''(w_i),\ i = 1, \ldots, n-1, \\ P_L S(w_n) &= \frac{r(w_n)}{24} h_{n-1}(5h_{n-1} - 4h_{n-2} + h_{n-3})Z, \end{aligned} \tag{5.5}$$

in which

$$Z = \frac{(h_{n-1} + h_{n-2}) \sqcap S''(w_{n-1}) - h_{n-1} \sqcap S''(w_{n-2})}{h_{n-2}}$$

and

$$\sqcap S''(w_j) = \frac{2h_j S''(w_{j-1}) - 2(h_{j-1} + h_j)S''(w_j) + 2h_{j-1}S''(w_{j+1})}{h_{j-1}(h_{j-1} + h_j)h_j}.$$

In the case of uniform partitioning ($h_i = h$), equation (5.5) becomes

$$\begin{aligned} P_L S(w_0) &= \frac{r(w_0)h^2}{12} \left[2 \sqcap S''(w_1) - \sqcap S''(w_2)\right] \\ P_L S(w_i) &= \frac{r(w_i)h^2}{12} \sqcap S''(w_i),\ i = 1, \ldots, n-1 \\ P_L S(w_n) &= \frac{r(w_n)h^2}{12} \left[2 \sqcap S''(w_{n-1}) - \sqcap S''(w_{n-2})\right] \end{aligned} \tag{5.6}$$

and

$$\sqcap S''(w_j) = \frac{1}{h^2} \left[S''(w_{j-1}) - 2S''(w_j) + S''(w_{j+1})\right]$$

The order of convergence of the optimal spline collocation method was shown to be of order four (see [7]).

6 Numerical Examples

To examine the accuracy and the local order of convergence of the proposed methods and to run comparison with other existing numerical methods, we chose four Bratu and Bratu-type equations whose exact solutions are known.

Example 6.1. Consider the Bratu boundary value problem

$$\begin{aligned} &u'' + 2e^u = 0,\ \ 0 < x < 1 \\ &u(0) = u(1) = 0 \end{aligned} \tag{6.1}$$

whose exact solution is obtained from (2.3)-(2.4) by letting $\lambda = 2$.

The optimal spline and cubic B-spline (B-spline) collocation methods were applied to this equation using uniform nodes. The absolute maximum errors and the local order of convergence for different uniform meshes, n, for the both methods are recorded in Table 2. The graphs of the approximate optimal spline solution and the exact solution for $n = 10$ are given in Figure 1.

Table 2
Numerical results for equation (6.1).

	B-spline		Optimal spline	
n	Max error	Order	Max error	Order
10	$8.83(-4)$		$2.64(-6)$	
20	$2.22(-4)$	2.0	$1.64(-7)$	4.0
40	$5.56(-5)$	2.0	$1.01(-8)$	4.0
80	$1.39(-5)$	2.0	$6.31(-10)$	4.0
160	$3.48(-6)$	2.0	$3.94(-11)$	4.0

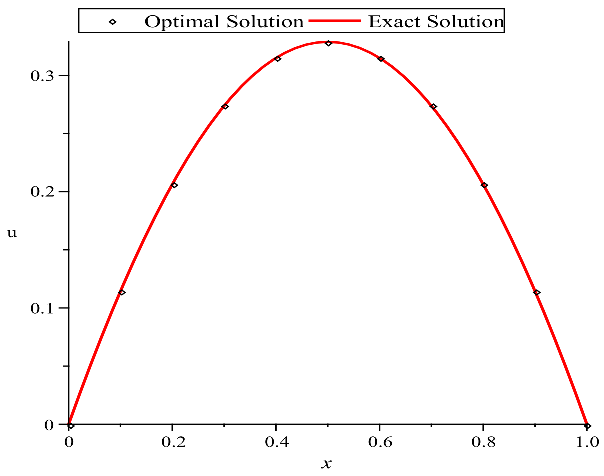

Figure 1. Exact and approximate optimal spline solutions with $n = 10$.

Example 6.2. Consider the Bratu-type boundary value problem

$$\begin{aligned} &u'' - \pi^2 e^{-u} = 0, \quad 0 < x < 1 \\ &u(0) = u(1) = 0 \end{aligned} \tag{6.2}$$

for which the exact solution is $u(x) = \ln(1 + \sin(1 + \pi x))$ (see [13]).

Equation (6.2) was solved by the cubic B-spline, the optimal spline, and the adaptive spline . In Table 3, the maximum error and the local order of convergence for different mesh size, n were recorded.

Table 3
Numerical results for equation (6.2).

	B-spline		Optimal spline		Adaptive spline	
n	Max error	Order	Max error	Order	Max error	Order
10	$2.60(-3)$		$1.14(-5)$		$1.14(-5)$	
20	$6.40(-4)$	2.0	$1.36(-6)$	3.1	$1.10(-6)$	3.4
40	$1.59(-4)$	2.0	$1.03(-7)$	3.7	$7.56(-8)$	3.9
80	$3.98(-5)$	2.0	$6.82(-9)$	3.9	$5.35(-9)$	3.8
160	$9.96(-6)$	2.0	$4.34(-10)$	4.0	$3.34(-10)$	4.0

Example 6.3. Consider the Bratu-type initial value problem

$$\begin{aligned} &u'' - 2e^{u} = 0, \quad 0 < x < 1 \\ &u(0) = u'(0) = 0 \end{aligned} \tag{6.3}$$

whose exact solution is $u(x) = -2\ln(\cos x)$.

The proposed optimal spline collocation method was compared to the perturbation-iteration algorithm with parameters $n = 1$ and $m = 3$, denoted PIA(1,3) (see [2] for details). Using only ten uniform mesh points ($n = 10$), Table 4 shows that the absolute error obtained by the optimal spline method is significantly smaller than that obtained the PIA(1,3). The graphs of the absolute errors obtained by the optimal spline method and the PIA(1,3) algorithm with $n = 1$ and $m = 3$ are given in Figure 2.

Table 4
Absolute errors using optimal spline and PIA algorithm [2] for equation (6.3).

x	Optimal spline	PIA$(1,3)$
0.1	7.25(−7)	6.71(−6)
0.2	1.61(−6)	9.55(−6)
0.3	2.38(−6)	3.31(−6)
0.4	3.02(−6)	8.04(−6)
0.5	3.45(−6)	8.48(−6)
0.6	3.51(−6)	2.03(−5)
0.7	2.78(−6)	7.15(−5)
0.8	7.06(−7)	2.91(−4)
0.9	6.86(−6)	1.05(−3)
1.0	3.28(−6)	3.53(−3)

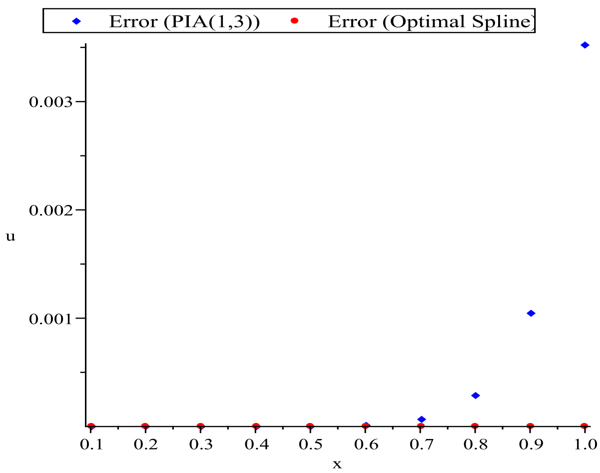

Figure 2. Absolute errors for the optimal spline and the PIA(1,2) with $n = 10$.

Example 6.4. Consider the Bratu boundary value problem

$$\begin{aligned} &u'' + e^u = 0, \ \ 0 < x < 1 \\ &u(0) = u(1) = 0 \end{aligned} \tag{6.4}$$

whose exact solution is obtained by substituting $\lambda = 1$ in (2.3).

The proposed optimal spline method and the perturbation iteration algorithm with $n = 1$, and $m = 2$ (PIA(1,2)) were carried out to find an approximate solution for equation (6.4). Using $n = 10$, the absolute error obtained the optimal spline method is shown in Table 5 to be significantly smaller than that of PIA(1,2) method for various x

values.

Table 5
Numerical results for equation (6.4).

x	Abs error for optimal spline	Abs error for PIA(1,2)
0.1	$4.63(-8)$	$1.68(-5)$
0.2	$1.02(-7)$	$3.99(-5)$
0.3	$1.44(-7)$	$4.91(-5)$
0.4	$1.71(-7)$	$6.03(-5)$
0.5	$1.81(-7)$	$5.92(-5)$
0.6	$1.71(-7)$	$6.03(-5)$
0.7	$1.44(-7)$	$4.91(-5)$
0.8	$1.02(-7)$	$3.99(-5)$
0.9	$4.63(-8)$	$1.68(-5)$

7 Conclusion

In addition to the cubic B-spline collocation method, an adaptive spline collocation, and an optimal collocation methods were derived to solve Bratu and Bratu-type equations. In order to be able to apply the adaptive spline collocation on nonuniform meshes, we began by the preliminary step, which is applying the iteration scheme resulting from implementing Newton's method and then choosing the appropriate grading function. The optimal spline collocation described by [6] was modified to suite the nonlinear Bratu-type boundary or initial value problems. Numerical examples were presented to show the applicability and efficiency of the proposed methods and their accuracies when compared to other existing numerical methods.

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