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MEAN VARIANCE OPTIMIZATION OF PORTFOLIOS

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Abstract. The Mean-Variance Portfolio Theory continues to be the cardinal tool for much of portfolio management. Traditional concerted literature on the Mean-Variance theory can be segmented almost exclusively into (i) chapters in books that provide simply a write up on the theory and (ii) books that contain a purely mathematical analysis without emphasizing the financial implications and interpretations. The fallout of this mutually exclusive segmentation is that both segments cover Mean- Variance portfolio theory only marginally i.e. in the asymptote rather than as a mainstream course. The coverage is, nowhere near adequate for a student to get acquainted with the intricacies of the theory and hence, appreciate its nuances. This article fights that trend by covering in detail the topics that are thrown by the wayside in the traditional coverage. We look at a comprehensive mathematical analysis of the two security problem in risk return space, obtain several interesting mathematical results and follow up each of them with their interpretation and explanations in financial markets. We, then, extend the framework to three security dynamics and again elucidate some intriguing mathematical inferences.

1 Introduction

The Mean-Variance portfolio theory continues to emerge as the cornerstone of modern portfolio management. The vital feature of the theory is its robustness just like the Economic Order Quantity (EOQ) model in Operational Research. The objective of portfolio management viz. the optimal allocation of the investments between available spectrum of assets is evaluated in a two dimensional risk-return framework. The “efficient frontier” in the Mean-Variance framework enables us to identify the optimal portfolio given a particular level of risk tolerance. It is emphasized, however, that the level of risk tolerance is singular to the investor’s risk profile and, as such, does not, strictly speaking, come within the domain of Mean-Variance portfolio theory – it may be handled by, for example, utility theory or the indifference map etc.

2 Concept of Risk & Return in the Mean- Variance Framework

In the Mean – Variance framework of portfolio management, we represent and evaluate securities in a two dimensional framework (i.e. risk and return) with, conventionally, the risk, being expressed along the abscissa (axis) and expected return along the ordinate (axis). In this context, instantaneous return is usually measured in terms of the accretion in the value of the security over an infinitesimal time period i.e.

$$dR(t) = \frac{dS(t)}{S(t)} \tag{2.1}$$

Correspondingly, the return over a finite time span $t_2 - t_1$ is given by

$$R(t_2, t_1) = \frac{S(t_2) - S(t_1)}{S(t_1)} \tag{2.2}$$

Eq. (2.2) shall, obviously need to be adjusted for any intermediate cash flows emanating from the security during the period $t_2 - t_1$. While the measure of return (2.2) is very convenient for measuring single period returns, extension to multi-period cases results in a very serious problem viz. the formula (2.2) is not additive. In other words,

$$R_{average}(t_2, t_0) = \frac{S(t_2) - S(t_0)}{2S(t_0)} \neq \frac{R(t_2, t_1) + R(t_1, t_0)}{2} \tag{2.3}$$

As a remedy to the problem of non-additivity, the concept of “logarithmic return” finds its way into the literature. It is defined by integrating eq. (2.1) and we obtain

$$R_{\ln}(t_2, t_1) = \log_e \frac{S(t_2)}{S(t_1)} \tag{2.4}$$

whence

$$R_{\ln, average}(t_2, t_0) = \frac{1}{2} \log_e \frac{S(t_2)}{S(t_0)} = \frac{1}{2} \left[\log_e \frac{S(t_2)}{S(t_1)} + \log_e \frac{S(t_1)}{S(t_0)} \right] = \frac{R_{\ln}(t_2, t_1) + R_{\ln}(t_1, t_0)}{2} \tag{2.5}$$

For small returns, the two measures are equivalent for

$$\log_e \frac{S(t_2)}{S(t_1)} = \log_e \left[1 + \frac{S(t_2) - S(t_1)}{S(t_1)}\right] \approx \frac{S(t_2) - S(t_1)}{S(t_1)} \tag{2.6}$$

"Risk" is, usually, interpreted as the "uncertainty" associated with an experiment in achieving its desired outcome i.e. the probability of the outcome of the experiment not being able to attain the target. Consequently, while evaluating risk, we are concerned with "downside" aspect of the return patterns i.e. the probability of actual returns falling short of targeted returns rather than "upside" or the probability of returns exceeding targets. However, in "Mean – Variance" Portfolio Theory, we measure risk in terms of the standard deviation of the security returns. There is some rationale behind this e.g.
(a) The "uncertainty" is directly related to the level of fluctuations or "dispersion" about the mean value i.e. higher the amplitude of swing about the mean value, higher is the uncertainty of achieving the targeted return;
(b) The return structure of securities is assumed symmetric so that the level of downside fluctuations equals the level of upside fluctuations;
(c) The stock process or the logarithm thereof is assumed to follow a normal distribution which is completely parameterized by the mean/expected value and standard deviation.

3 Return and Variance of a Portfolio of Securities

The instantaneous, expected returns and variances of a portfolio of N securities with composition vector $X = \{X_i, i = 1, 2, 3, ..., N\}$, $\sum_{i=1}^{N} X_i = 1$ are given respectively by:

$$R_P = \sum_{i=1}^{N} X_i R_i \tag{3.1}$$

$$E(R_P) = \sum_{i=1}^{N} X_i E(R_i) \tag{3.2}$$

$$\sigma_P^2 = E\left[R_P - E(R_P)\right]^2 = \sum_{i=1}^{N}\sum_{j=1}^{N} X_i X_j \sigma_{ij} = \sum_{i=1}^{N} X_i^2\sigma_i^2 + \sum_{i=1}^{N} \sum_{\substack{j=1 \\ i \neq j}}^{N} X_i X_j \sigma_{ij} = \sum_{i=1}^{N} X_i^2\sigma_i^2 + 2\sum_{i=1}^{N} \sum_{\substack{j=1 \\ i < j}}^{N} X_i X_j \sigma_{ij} \tag{3.3}$$

4 The Portfolio Possibilities Curve (PPC) for Two Security Portfolio

We define the portfolio possibilities curve (PPC) as the locus of a point in risk-return space that identifies an admissible portfolio. For a two security portfolio with composition vector $X = \{X_1, 1 - X_1\}$, we have, from eqs. (3.2), (3.3) with $\rho = \sigma_{12}\sigma_1^{-1}\sigma_2^{-1}$

$$E(R_P) = X_1 E(R_1) + (1 - X_1) E(R_2) \tag{4.1}$$

$$\sigma_P^2 = X_1^2\sigma_1^2 + (1 - X_1)^2 \sigma_2^2 + 2X_1(1 - X_1)\rho\sigma_1\sigma_2 \tag{4.2}$$

Eliminating X_1 between eqs. (4.1) & (4.2), we obtain the equation for the PPC for the two security case as:

$$x^2 - y^2 \frac{\left(\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2\right)}{(R_1 - R_2)^2} + 2y \frac{\left[R_2\sigma_1^2 + R_1\sigma_2^2 - (R_1 + R_2)\rho\sigma_1\sigma_2\right]}{(R_1 - R_2)^2} - \frac{\left(R_2^2\sigma_1^2 + R_1^2\sigma_2^2 - 2R_1R_2\rho\sigma_1\sigma_2\right)}{(R_1 - R_2)^2} = 0 \tag{4.3}$$

where we have abbreviated $E(R_P) \equiv y, \sigma_P = x, E(R_1) \equiv R_1, E(R_2) \equiv R_2$.
Comparing eq. (4.3) with the general equation of a conic viz.

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \tag{4.4}$$

we obtain $a = 1, h = 0, b = -\frac{\left(\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2\right)}{(R_1-R_2)^2}$ whence $h^2 - ab > 0$ so that the PPC represents a hyperbola in shape.
The equation of the PPC can be written as

$$x^2 - by^2 + 2fy - c = 0 \;\; or \;\; \frac{x^2}{c - \frac{f^2}{b}} - \frac{\left(y\sqrt{b} - \frac{f}{\sqrt{b}}\right)^2}{c - \frac{f^2}{b}} = 1 \tag{4.5}$$

where

$$b = \frac{\left(\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2\right)}{(R_1 - R_2)^2}, f = \frac{\left[R_2\sigma_1^2 + R_1\sigma_2^2 - (R_1 + R_2)\rho\sigma_1\sigma_2\right]}{(R_1 - R_2)^2}, c = \frac{\left(R_2^2\sigma_1^2 + R_1^2\sigma_2^2 - 2R_1R_2\rho\sigma_1\sigma_2\right)}{(R_1 - R_2)^2} \tag{4.6}$$

It is important at this point to note the following:
(a) $x \equiv \sigma$ represents standard deviation of a random variable (security returns) and hence, can never be negative;
(b) Assuming **no short sales**, the portfolio return $y \equiv R_P$ must necessarily lie between R_1 & R_2 so that no point of the PPC can lie outside the region bounded by the abscissa through R_1 & R_2;
(c) We must also have$-1 \leq \rho \leq 1$. Let us,, now, examine these two extremal cases:

For perfectly correlated assets, $\rho = +1$, eq. (4.3) becomes

$$y = \frac{(R_1 - R_2)}{(\sigma_1 - \sigma_2)} x + \frac{(R_2\sigma_1 - R_1\sigma_2)}{(\sigma_1 - \sigma_2)} \tag{4.7}$$

which is a straight line with gradient $\frac{(R_1 - R_2)}{(\sigma_1 - \sigma_2)}$, intercept on the ordinate axis $\frac{(R_2\sigma_1 - R_1\sigma_2)}{(\sigma_1 - \sigma_2)}$ and passing through the points$A(\sigma_1, R_1)$ &$B(\sigma_2, R_2)$ representing the two securities in risk-return space. Hence, any portfolio of two perfectly correlated securities will lie on the straight line joining the two securities in risk-return space and the PPC, in this case, is the straight line joining these two points. The case of anti-correlated assets $(\rho = -1)$ is relatively more involved. The eq. of the PPC becomes

$$(R_1 - R_2)\, x = \pm \left[(\sigma_1 + \sigma_2)\, y - (R_1\sigma_2 + R_2\sigma_1)\right] \tag{4.8}$$

Since $x$being standard deviation must necessarily be positive, the sign of the RHS is dictated by the sign of $(R_1 - R_2)$ so that we shall have two scenarios and hence, a pair of straight lines
(i) in the region where $sgn\,(R_1 - R_2) = sgn\left[(\sigma_1 + \sigma_2)\, y - (R_1\sigma_2 + R_2\sigma_1)\right]$ the positive sign outside the square bracket will hold and the equation of the PPC in this region will be

$$y = \frac{(R_1 - R_2)}{(\sigma_1 + \sigma_2)} x + \frac{(R_1\sigma_2 + R_2\sigma_1)}{(\sigma_1 + \sigma_2)} \tag{4.9}$$

(ii) in the region where $sgn\,(R_1 - R_2) \neq sgn\left[(\sigma_1 + \sigma_2)\, y - (R_1\sigma_2 + R_2\sigma_1)\right]$, the negative sign outside the square bracket will hold and the equation of the PPC in this region will be

$$y = -\frac{(R_1 - R_2)}{(\sigma_1 + \sigma_2)} x + \frac{(R_1\sigma_2 + R_2\sigma_1)}{(\sigma_1 + \sigma_2)} \tag{4.10}$$

In fact, by an appropriate numbering of the two securities, we can always ensure that $(R_1 - R_2) \geq 0$ whence eq. (4.9) will operate in the region where $y \geq \frac{(R_1\sigma_2 + R_2\sigma_1)}{(\sigma_1 + \sigma_2)}$ or equivalently $X_1 \geq \frac{\sigma_2}{(\sigma_1 + \sigma_2)}$ and eq. (4.10) in the region where $X_1 \leq \frac{\sigma_2}{(\sigma_1 + \sigma_2)}$. It is pertinent to mention that the straight lines (4.9) & (4.10) intersect each other and the ordinate axis at the point $F\left(0, \frac{(R_1\sigma_2 + R_2\sigma_1)}{(\sigma_1 + \sigma_2)}\right)$ which identifies the risk free rate of return. Further, eq. (4.9) is the join of the point F& A while (4.10) is the join of F& B so that for the entire range of values $0 \leq X_1 \leq 1$, the risk free ordinate $F\left(0, \frac{(R_1\sigma_2 + R_2\sigma_1)}{(\sigma_1 + \sigma_2)}\right)$ is unique. Needless to add, the return $R_F = \frac{(R_1\sigma_2 + R_2\sigma_1)}{(\sigma_1 + \sigma_2)}$ lies betweenR_1, R_2.
The fallout of the observations above is that the PPC shall be confined to the section of the hyperbola lying in the first quadrant between the lines given by eqs. (4.7), (4.9) & (4.10) that, incidentally form a triangle with the vertices $A(\sigma_1, R_1)$, $B(\sigma_2, R_2)$ and $F\left(0, \frac{(R_1\sigma_2 + R_2\sigma_1)}{(\sigma_1 + \sigma_2)}\right)$. The exact shape of the hyperbola is parameterized by the correlation coefficient between the two given securities,ρ.

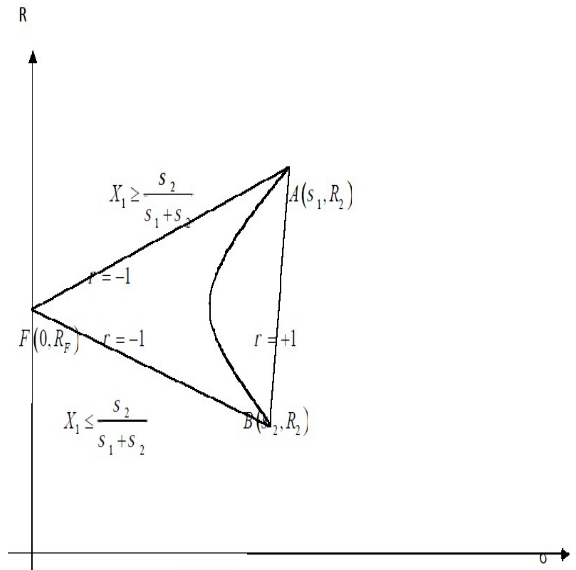

Figure 1

It is instructive to calculate the circumstances under which a riskless portfolio can be constructed from two risky assets. For the purpose, the PPC must intersect the ordinate axis at real points. In other words, the intersection of eq. (4.3) with $x = 0$ i.e.

$$y^2\frac{\left(\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2\right)}{\left(R_1-R_2\right)^2}-2y\frac{\left[R_2\sigma_1^2+R_1\sigma_2^2-\left(R_1+R_2\right)\rho\sigma_1\sigma_2\right]}{\left(R_1-R_2\right)^2}+\frac{\left(R_2^2\sigma_1^2+R_1^2\sigma_2^2-2R_1R_2\rho\sigma_1\sigma_2\right)}{\left(R_1-R_2\right)^2}=0 \quad (4.11)$$

must have real roots, the condition for which, on simplification, is found to be

$$\sigma_1^2\sigma_2^2\left(R_1-R_2\right)^2\left(\rho^2-1\right)\geq 0 \quad (4.12)$$

yielding $\rho=\pm 1$ so that a risk free asset can be constructed out of two risky assets only if they are perfectly (anti) correlated. The case of perfectly correlated assets can yield a risk free asset only in the circumstances when short sales are permitted. This is seen from eq,. (4.7). The ordinate intercept in that case is given by $R_F = \frac{(R_2\sigma_1-R_1\sigma_2)}{(\sigma_1-\sigma_2)}$ so that $R_F<\min\left(R_1,R_2\right)$ or$R_F>\max\left(R_1,R_2\right)$. For $R_P=X_1R_1+\left(1-X_1\right)R_2=R_F=\frac{(R_2\sigma_1-R_1\sigma_2)}{(\sigma_1-\sigma_2)}$, we obtain $X_1=\frac{\sigma_2}{\sigma_2-\sigma_1}<0$ (assuming $\sigma_1>\sigma_2$) implying short sales of security A since $X_1<0\Rightarrow X_2>1$.

5 Tracing the Portfolio Possibilities Curve

The equation of the PPC is given by eq. (4.3) or eq. (4.5). Salient characteristics of the curve are listed below:
(a) **Asymptotes**: The pair of asymptotes to the curve (4.5) are given by:

$$\frac{x^2}{c-\frac{f^2}{b}}-\frac{\left(y\sqrt{b}-\frac{f}{\sqrt{b}}\right)^2}{c-\frac{f^2}{b}}=0=x^2-\left(y\sqrt{b}-\frac{f}{\sqrt{b}}\right)^2 \quad (5.1)$$

or equivalently by

$$y=\pm\frac{x}{\sqrt{b}}+\frac{f}{b} \quad (5.2)$$

written out explicitly as

$$y=\pm\frac{\left(R_1-R_2\right)x}{\sqrt{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}}+\frac{\left[R_2\sigma_1^2+R_1\sigma_2^2-\left(R_1+R_2\right)\rho\sigma_1\sigma_2\right]}{\left(\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2\right)} \quad (5.3)$$

It is pertinent to note that for the two extremal cases $\rho=\pm 1$, the PPC coincides with its asymptotes.
(b) **Axes**: The axes of the PPC are respectively

$$x=0, y=\frac{f}{b}=\frac{\left[R_2\sigma_1^2+R_1\sigma_2^2-\left(R_1+R_2\right)\rho\sigma_1\sigma_2\right]}{\left(\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2\right)} \quad (5.4)$$

(c) **Coordinates of the Centre**:

$$x=0, y=\frac{f}{b}=\frac{\left[R_2\sigma_1^2+R_1\sigma_2^2-\left(R_1+R_2\right)\rho\sigma_1\sigma_2\right]}{\left(\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2\right)} \quad (5.5)$$

(d) **Point of Inflexion**: From eq. (4.5), we have $\frac{dx}{dy}=\frac{by-f}{x}$. For the point of inflexion, we set $\frac{dx}{dy}=0$ whence

$$y_{\text{inflexion}}=\frac{f}{b}=\frac{\left[R_2\sigma_1^2+R_1\sigma_2^2-\left(R_1+R_2\right)\rho\sigma_1\sigma_2\right]}{\left(\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2\right)} \quad (5.6)$$

The corresponding abscissa is

$$x_{\text{inflexion}}=\pm\sqrt{\left(by^2-2fy+c\right)}=\pm\sqrt{\left[b\left(\frac{f}{b}\right)^2-2f\left(\frac{f}{b}\right)+c\right]}=\pm\sqrt{\left(c-\frac{f^2}{b}\right)}$$
$$=\pm\left[\frac{\left(1-\rho^2\right)\sigma_1^2\sigma_2^2}{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}\right]^{1/2} \quad (5.7)$$

We shall show in the sequel that this point of inflexion corresponds to point of minimum variance.

6 The Minimum Variance Portfolio

The composition of the Minimum Variance portfolio, M is obtained by differentiating eq. (4.2) with respect to X_1 and equating to zero whence we obtain

$$X_M=\left(\frac{\sigma_2^2-\rho\sigma_1\sigma_2}{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2},\frac{\sigma_1^2-\rho\sigma_1\sigma_2}{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}\right) \quad (6.1)$$

and the coordinates of$M$in risk-return space are obtained by substituting this composition vector in eqs. (4.1) & (4.2). They are found to be

$$\sigma_M = \left[\frac{(1-\rho^2)\,\sigma_1^2\sigma_2^2}{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}\right]^{1/2} \tag{6.2}$$

$$R_M = \frac{\left[R_2\sigma_1^2 + R_1\sigma_2^2 - (R_1+R_2)\,\rho\sigma_1\sigma_2\right]}{\left(\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2\right)} \tag{6.3}$$

showing that the point of inflexion of the PPC coincides with M. The locus of M is found by eliminating ρ between eqs. (6.2) & (6.3) and we have

$$\left[\frac{\left(R_1\sigma_2^2 + R_2\sigma_1^2\right) - y\left(\sigma_1^2+\sigma_2^2\right)}{(R_1+R_2)-2y} - x^2\right]^2 = \left(x-\sigma_1^2\right)\left(x-\sigma_2^2\right) \tag{6.4}$$

where $\sigma_M \equiv x, R_M = y$.

7 The PPC with Short Sales Permitted

In the event when short sales are permitted, the components of the composition vector become unbounded $-\infty < X_i < \infty$ with the only constraint $X_1 + X_2 = 1$. Hence, we can create portfolios with unbounded positive as well as negative returns (hypothetically) by short selling one or the other asset and investing the proceeds on the second asset. In such a case, the PPC gets extended beyond A, B along the same hyperbola i.e. the PPC consists of the entire section of the hyperbola that lies in the right hand side half plane bounded by the Y axis.

8 The PPC with one of Securities being Riskfree

Let the asset A, renamed Fbe a riskfree asset so that $\sigma_1 = \rho = 0$, $R_1 = R_F$ so that eq. (4.3) for the PPC becomes

$$y = \pm\frac{R_2 - R_F}{\sigma_2}x + R_F \tag{8.1}$$

which is a pair of straight lines that intersect each other and the Yaxis at the point $F\,(0, R_F)$. Since$x$, being standard deviation must necessarily be positive, we have the following situation:
(a) If $R_2 - R_F > 0$, then the positive sign holds in eq. (8.1) in the region where $y - R_F > 0$ which corresponds to $X_F < 1$ i.e. no short sales of the risky security B. The PPC is the line segment FB terminating at the point $B\,(\sigma_2, R_2)$. The negative sign shall hold in the region where $y - R_F < 0$ corresponding to $X_F > 1$ that represents short sales of B and investment of the proceeds in the riskfree asset. With the possibility of unlimited short selling of B and investment of proceeds in F, the PPC in this case is the ray originating from F and extending to infinity with a slope that is the mirror image ofFB. If $R_2 - R_F < 0$, the converse will hold i.e. the negative sign holds in eq. (8.1) in the region where $y - R_F > 0$ and vice versa.
(b) Let short sales of the riskfree asset i.e. riskfree borrowing be permitted, so that $X_F < 0$ becomes admissible. With the potential possibility of unlimited riskless borrowing and investing in the risky asset, the PPC, in this case does not terminate at the point $B\,(\sigma_2, R_2)$but extends beyond B indefinitely along the straight line FB.

9 The PPC with two risky securities and a riskfree security

Let $A\,(\sigma_1, R_1)$ & $B\,(\sigma_2, R_2)$ be two risky securities and $F\,(0, R_F)$ be a riskfree security.
(a) In the case when short sales is not permitted in either of the two risky securities and riskless borrowing is also not allowed, the PPC takes the form of a surface bounded by the straight line segments AF, $BF$and the arc of the hyperbola ACB. The line segmentAFwill represent combinations of A and $F$with B being absent and BFwill represent combinations of B and $F$with A being absent. The arc of the hyperbola ACB will represent combinations of A and B exclusively. Any line segment CF will be a combination of all the three securities A,B & $F$where the relative proportion of A and B shall be determined by the location of C on ACB and that of F on the position of the portfolio point on CF.
It is pertinent to mention here that both straight line segments AF, BF shall intersect the closed arc ACB at no points other than A and B. This follows from (i) the point A must lie on the arc ACB since this arc represents portfolios of A and B that includes the portfolio of A alone and (ii) let, if possible, AF intersect ACB at another point D. Now, all points on the line segment AF must necessarily consist of only A and F. However, the security represented by the point$D$, that is common to ACB and AF shall consist of all the three securities, which is a contradiction.
(b) When short sales is permitted in A and B and riskless borrowing is not allowed, the PPC is determined as follows. We construct tangents from the point F to the arc of the hyperbolaACB, extended beyond A and B, if required. Let these tangents meet the extended arc of the hyperbola ACB at the points P and Q. The PPC, then consists of (i) the region PFQ being bounded by the straight line segmentsPF, $QF$and the arc PCQ (ii) the points on the arc of the hyperbola beyondCP, CQ extended indefinitely. The region PFA will represent combinations of the riskfree

asset (long) with the asset$A$ (long) and B (short) and similarly QFB will include A (short), B(long) together with the riskfree asset (long). Points within the region AFB will consist of combinations that are long in all the three securities. Points on CP, CQ extended beyond A, B respectively shall represent combinations of only A (long), B(short) and vice versa.

(c) When short sales is permitted in A and B and riskless borrowing is also allowed, the PPC is determined as in (b) above by constructing tangents from the point F to the arc of the hyperbolaACB, extended beyond A and B, if required intersecting ACB at the points P and Q. The PPC, then consists of the entire region of the positive X half plane that lies within the straight lines PF, QF extended indefinitely. In addition to the combinations explained in (b) above, points to the right of the arcPCQ shall contain riskless borrowing in addition to A and B while points lying in the region between the arc CP (extended) and FA (extended) beyond A represent combinations of riskless borrowing together with A (long) and B (short).

The coordinates of P and Q can be obtained in any of the following two ways:

(i) Let $y = mx + R_F$ be tangent to the hyperbola (4.5) so that it intersects the hyperbola at two coincident points, the condition for which is that the quadratic equation $x^2 - b(mx + R_F)^2 + 2f(mx + R_F) - c = 0$must have equal roots which gives $m = \pm\sqrt{\frac{bR_F^2 - 2fR_F + c}{bc - f^2}}$ whence the equation of the two tangents is

$$y = \pm x\sqrt{\frac{bR_F^2 - 2fR_F + c}{bc - f^2}} + R_F \tag{9.1}$$

and the coordinates of P and Q are respectively given by

$$\left(\frac{m(bR_F - f)}{1 - bm^2}, \pm\frac{m^2(bR_F - f)}{1 - bm^2} + R_F\right) \tag{9.2}$$

(ii) The second method makes use of the fact that the tangents PF, QF maximize $\tan\theta = \frac{R_P - R_F}{\sigma_P}$. Making use of eqs. (4.1) & (4.2), we obtain

$$\tan\theta = \frac{X_1(R_1 - R_F) + X_2(R_2 - R_F)}{\left[X_1^2\sigma_1^2 + X_2^2\sigma_2^2 + 2X_1X_2\rho\sigma_1\sigma_2\right]^{1/2}} \tag{9.3}$$

Taking partial derivatives, with respect to X_1, X_2 and equating them to zero, writing $\frac{R_P - R_F}{\sigma_P^2} = \lambda$, $Z_k = \lambda X_k$, $Z_1 + Z_2 = \lambda$, we obtain the following eqs. for the composition vector:

$$R_1 - R_F = Z_1\sigma_1^2 + Z_2\rho\sigma_1\sigma_2 \tag{9.4}$$

$$R_2 - R_F = Z_2\sigma_1^2 + Z_1\rho\sigma_1\sigma_2 \tag{9.5}$$

which can be solved to obtain the composition vector Xwhence we can obtain the coordinates of P and Q.

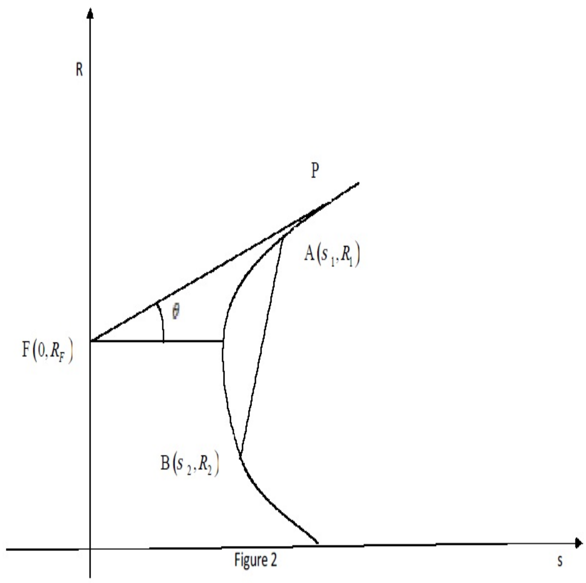

Figure 2

10 Concept of "Efficient Frontier"

To introduce the concept, we consider, first, the case of "no" short sales. Let $x = k$be any line ||Y-axis. Its intercepts with the PPC of eq. (4.3) are obtained by solving

$$k^2 - y^2\frac{\left(\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2\right)}{\left(R_1-R_2\right)^2} + 2y\frac{\left[R_2\sigma_1^2+R_1\sigma_2^2-\left(R_1+R_2\right)\rho\sigma_1\sigma_2\right]}{\left(R_1-R_2\right)^2}$$

$$-\frac{\left(R_2^2\sigma_1^2+R_1^2\sigma_2^2-2R_1R_2\rho\sigma_1\sigma_2\right)}{\left(R_1-R_2\right)^2} = 0 \tag{10.1}$$

This is a quadratic in$y$. For equal roots, we must have,

$$k^2 = \frac{\left(1-\rho^2\right)\sigma_1^2\sigma_2^2}{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2} = \sigma_M^2 \tag{10.2}$$

showing that there is only one point such that the straight line ||Y-axis is tangent to the PPC. Incidentally, this point coincides with the minimum variance point and the point of inflexion. Any other line ||Y-axis shall intersect the PPC at two distinct points (real or imaginary). We are concerned here only with real points. Then, the segment of the PPC that lies between the point of minimum variance $M$and A (assuming $R_1 > R_2$) superordinates over the segment of the PPC lying between M and Bin the sense that corresponding to every point on MB there exists a point on MAthat provides a higher return for the same level of risk. Thus, the portion of the arc MA dominates over the portion MB and, hence, is called the "efficient frontier". The "efficient frontier" corresponding to various scenarios discussed above is tabulated here:

Scenario	**Efficient Frontier**
Two risky assets, no short sales, no riskfree asset	The arc of the hyperbola lying between the minimum variance point $M$and A (assuming $R_1 > R_2$)
Short sales allowed, no risky asset	The arc of the hyperbola from the minimum variance point $M$and extending through A (assuming $R_1 > R_2$) indefinitely
One risky asset with riskless lending	The straight line joining the riskfree asset and the risky asset in risk-return space
One risky asset and riskless lending & borrowing	The straight line extending from the riskfree asset through the risky asset to infinity in risk-return space
Two risky assets, no short sales, riskfree lending	The straight line segment joining the riskfree asset to the risky asset with higher return in risk-return space.
Two risky assets, short sales allowed, riskless lending	The straight line segment from the riskfree asset and tangent, with positive slope, to the arc of the hyperbola upto the point of contact. From that point on, the arc of the hyperbola that represents combinations of the two risky assets only.
Two risky assets, short sales allowed, riskless lending & borrowing allowed	The straight line segment from the riskfree asset and tangent, with positive slope, to the arc of the hyperbola extended indefinitely.

11 The Case of Three Risky Securities

In the case of two risky securities, the problem of tracing out the PPC is relatively simple because of its immediate compatibility with the two dimensional framework. However, an analysis of the three securities PPC elucidates some intriguing features of the portfolio optimization problem. We shall illustrate these features by means of an example to avoid getting lost in a plethora of calculations.

For the purpose, we consider three risky securities$A\left(6,14\right)$, $B\left(3,8\right)$ and $C\left(15,20\right)$ with $\rho_{AB} = 0.50$,$\rho_{BC} = 0.40$ and $\rho_{CA} = 0.20$ with the composition vector $X = \{X_1, X_2, X_3\} \equiv \{1-X_2-X_3, X_2, X_3\}$. The equation of the PPC is obtained e.g. in terms of $x \equiv \sigma_P$, $y \equiv R_P$ and $z \equiv X_3$ by eliminating X_2 between the equations for expected return and standard deviation given by eqs. (3.2) & (3.3) and we obtain

$$x^2 - \frac{3}{4}y^2 - 306z^2 + 12y - 162z + \text{`}18yz - 57 = 0 \tag{11.1}$$

It is easily seen that the projection of the above curve on each of the three planes is a hyperbola. However, we need to focus on the XY plane. We can write eq. (11.1) as

$$\frac{x^2}{198z^2+18z+9}-\frac{\left(\sqrt{\frac{3}{4}}y-\frac{18z+12}{\sqrt{3}}\right)^2}{198z^2+18z+9}=1 \tag{11.2}$$

(a) **Asymptotes**: The pair of asymptotes to the curve (11.2) are given by:

$$y=\pm\frac{2x}{\sqrt{3}}+12z+8 \tag{11.3}$$

(b) **Axes**: The axes of the PPC are respectively

$$x=0, y=12z+8 \tag{11.4}$$

(c) **Coordinates of the Centre**:

$$x=0, y=12z+8 \tag{11.5}$$

(d) **Point of Inflexion**: The point of inflexion is given by

$$x=\pm\sqrt{198z^2+18z+9}, y=12z+8 \tag{11.6}$$

The above characteristics reveal that the projection of the PPC on the XY plane shall consist of a family of hyperbole $\{\mathrm{H}_z\}$with each hyperbola corresponding to a value of $z\equiv X_3$. The centre of the hyperbola moves up along the Yaxis as more of the security $C\,(15,20)$ is inducted into the portfolio and the point of inflexion also moves away from the abscissa as well as the ordinate axes showing that the minimum variance portfolio increases both in terms of the expected return and variance. Hence, the portfolio optimization problem, in essence, boils down to (i) identifying that hyperbola out of the family (of hyperbole$\{\mathrm{H}_z\}$) which is such that the value of $\tan\theta=\frac{R_P-R_F}{\sigma_P}$ i.e. slope of the tangent drawn from the riskfree asset to the hyperbola is maximum. Let this hyperbola be H_α; (ii) once the hyperbola is identified, to obtain the coordinates of the point of contact of the tangent (that has the maximum slope) with the hyperbola H_α (to which it is tangent). The efficient frontier then simply becomes the straight line joining the riskfree asset with the point of contact.

The procedure is purely an extension of the one set out in Section 7(c)(ii). Since a generalization to the N securities is absolutely straight forward, we set out the procedure for the latter, in view of its practical importance. Setting $\tan\theta=\frac{R_P-R_F}{\sigma_P}$, making use of eqs. (4.1) & (4.2), we obtain

$$\tan\theta=\frac{\sum_{i=1}^{N}X_i\left(R_i-R_F\right)}{\left[\sum_{i=1}^{N}X_i^2\sigma_i^2+\sum_{i=1}^{n}\sum_{\substack{j=1\\ i\neq j}}^{n}X_iX_j\sigma_{ij}\right]^{1/2}} \tag{11.7}$$

Taking partial derivatives, with respect to each X_i and equating them to zero, writing $\frac{R_P-R_F}{\sigma_P^2}=\lambda$, $Z_k=\lambda X_k$, $\sum_{i=1}^{N}Z_i=\lambda$, we obtain the following eqs. for the composition vector:

$$R_i-R_F=Z_i\sigma_i^2+\sum_{j=1,j\neq i}^{N}Z_j\sigma_{ij}, i=1,2,3,...,N \tag{11.8}$$

Thus, we get a set of N equations for an equal number of unknowns, being the components of the composition vector $X=\{X_i, i=1,2,3,...,N\}$which would, in the normal course, have a unique solution corresponding to the point of contact of the tangent to the hyperbola $\mathrm{H}_\alpha$identified as above. Knowing the composition vector, it is rudimentary to calculate the corresponding coordinates in risk-return space. The point so obtained would be the point of contact of the tangent of greatest slope with the hyperbola H_α. The efficient frontier is then, the straight line joining the riskfree asset with this point, extended to infinity, if riskless borrowing is permitted.

The final question is, what happens when neither riskless borrowing nor lending is permitted, only short sales of the risky securities is allowed? What would be the efficient frontier and how do we trace it?

The efficient frontier, in that case is the arc of the hyperbola H_α extending from the point of minimum variance on H_α upwards to infinity. To trace out the arc, we may follow the following:

(i) Let the point of contact of the maximum slope tangent with the hyperbola H_α be designated P and let the corresponding composition vector be X_P. Taking a different riskfree rate, say, R_F' and solving the set of equations (11.8) corresponding to R_F' we obtain Q (with composition vector X_Q) that is the point of contact of the tangent with maximum slope with one of the hyperbole of the family H_z. Now, it turns out that the hyperbola H_α is the optimal hyperbola of the family H_z for all the riskfree rates R_F so that the point Q also lies on H_α.

(ii) Let us construct a third portfolio R that is the average of both P & Q so that $X_R = 0.50\,(X_P + X_Q)$. Knowing the composition vector X_R, we can straight away calculate the standard deviation $\sigma_R$using eq. (3.3) e.g.

$$\sigma_R^2 = \sum_{i=1}^{N}\sum_{j=1}^{N} X_i^R X_j^R \sigma_{ij} = 0.25 \sum_{i=1}^{N}\sum_{j=1}^{N} \left[\left(X_i^P + X_i^Q\right)\left(X_j^P + X_j^Q\right)\right]\sigma_{ij} \tag{11.9}$$

(iii) Treating P & Q as separate securities and knowing their respective variances, we can express the variance of R as

$$\sigma_R^2 = 0.25\left(\sigma_P^2 + \sigma_Q^2 + 2\rho_{PQ}\sigma_P\sigma_Q\right) \tag{11.10}$$

where ρ_{PQ} is, as yet, unknown. However, by equating the two expressions for $\sigma_R$given by eqs. (11.9) & (11.10), we can obtain ρ_{PQ} whence the problem of tracing the efficient frontier (i.e. the extended arc of the hyperbola PQ from the point of minimum variance) gets reduced to a two-security problem and can be easily solved as given in the earlier sections of this work. It may be noted that all the relevant parameters of both P & Q viz. their variances, covariance and expected returns are known quantities now.

References

1. Bawa, V, Elton, E J, Gruber, M J, Journal of Finance, 34, No. 2, 1979;
2. Bodie, Z, Kane, A & Marcus, A J, Investments, Irwin, 2008;
3. Chen, S, and Brown, S J, Journal of Finance, 38, No. 4, 1983;
4. Dowd, K, Measuring Market Risk, John Wiley, 2005;
5. Duffue, D & Duffie, J D, Dynamic Asset Pricing Theory, placePrinceton, 2001;
6. Elton, E J & Gruber, M J, Modern Portfolio Theory & Investment Analysis, John Wiley, 2006;
7. Elton, E J, Gruber, M J & Padberg, M W, Journal of Finance, XI, No. 5, 1976;
8. Elton, E J, Gruber, M, J & Padberg, M W, Journal of Finance, XIII, No. 1, 1978;
9. Farrell, J L, & Reinhart, W J, Portfolio Management, Theory & Application, McGraw Hill, 1996;
10. Green, R C & Hollifield, B, Journal of Finance, 47, No. 5, 1992;
11. Haugen, R A, Modern Investment Theory, Printice Hall, 1990;
12. Mcdonnell, P J, Optimal Portfolio Modeling, John Wiley, 2008;
13. Reilly, F K & Brown, KC, Investment Analysis & Portfolio Management, Thomson Learning, 2006;
14. Strong R A, Portfolio Construction, Management & Protection, Thomson Learning, 2005;

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