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On Lie ideals with symmetric bi-additive maps in rings

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Abstract. Let R be a ring and $U \neq 0$ be a Lie ideal of R. A bi-additive symmetric map $B(.,.) : R \times R \to R$ is called symmetric bi-derivation if, for any $y \in R$, the map $x \mapsto B(x, y)$ is a derivation. A mapping $f : R \to R$ defined by $f(x) = B(x, x)$ is called the trace of B. In the present paper, we shall show that $U \subseteq Z(R)$ such that R is a prime and semiprime ring admitting the trace f satisfying the several conditions of symmetric bi-derivation.

1 Introduction

Throughout this paper, all rings will be associative. The center of a ring R will be denoted by $Z(R)$. Recall that a ring R is prime if $aRb = \{0\}$ implies $a = 0$ or $b = 0$ and semiprime in case $aRa = \{0\}$ implies $a = 0$. For any $x, y \in R$, the symbol $[x, y]$ will represent the commutator $xy - yx$ and the symbol $x \circ y$ stands for the anti-commutator (or skew-commutator) $xy + yx$. An additive mapping $d : R \to R$ is called derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. A derivation d is inner if there exists a fixed $a \in R$ such that $d(x) = [a, x]$ holds for all $x \in R$. A mapping $A(.,.)R \times R \to R$ is said to be symmetric if $A(x, y) = A(y, x)$ for all $x, y \in R$. A mapping $f : R \to R$ defined by $f(x) = A(x, x)$, where $A(.,.) : R \times R \to R$ is symmetric mappings, is called the trace of A. It is obvious that, in case $A(.,.) : R \times R \to R$ is a symmetric mapping which is also a bi-additive (i.e., additive in both arguments). The trace of A satisfies the relation $f(x + y) = f(x) + f(y) + 2A(x, y)$ for all $x, y \in R$.

A symmetric bi-additive mapping $B(.,.) : R \times R \to R$ is called a symmetric bi-derivation if $B(xy, z) = B(x, z)y + xB(y, z)$ for all $x, y, z \in R$. The concept of symmetric bi-derivation was introduced by G. Maksa [7] (see also [6] where an example can be found).

A study on the theory of centralizing (commuting) maps on prime rings was initiated by the classical result of Posner [9] which stated that the existence of a nonzero centralizing derivation on a prime ring R implies that R is commutative. Since then, a great deal of work in this context has been done by the number of authors (see, e.g., [1], [3] and references therein). For example, as a study concerning centralizing (commuting) maps, Vukman [10],[11] investigated symmetric bi-derivations in prime and semiprime rings. In [1] Argec and Yenigul and Muthana [8] obtained the similar type of results on Lie ideals of R. The objective of this paper is to study the commutativity of prime and semiprime rings satisfying various identities involving the trace f of the symmetric bi-derivation B. In fact we obtain rather more general results by considering various conditions on a subset of the ring R viz. Lie ideal of R.

2 Preliminaries

We shall frequently use the following identities and several well known facts about the semiprime ring without specific mention.

(1) $[xy, z] = x[y, z] + [x, z]y$

(2) $[x, yz] = y[x, z] + [x, y]z$

(3) $x \circ yz = (x \circ y)z - y[x, z] = y(x \circ z) + [x, y]z$

(4) $(xy) \circ z = x(y \circ z) - [x, z]y = (x \circ z)y + x[y, z]$.

Remark 2.1. Let U be a square closed Lie ideal of R. Notice that $xy + yx = (x + y)^2 - x^2 - y^2$ for all $x, y \in U$. Since $x^2 \in U$ for all $x \in U$, $xy + yx \in U$ for all $x, y \in U$. Hence we find that

$2xy \in U$ for all $x, y \in U$. Therefore, for all $r \in R$, we get $2r[x,y] = 2[x,ry] - 2[x,r]y \in U$ and $2[x,y]r = 2[x,yr] - 2[y,r] \in U$, so that $2R[U,U] \subseteq U$ and $2[U,U]R \subseteq U$.

This remark will be freely used in the whole paper without specific reference.

Lemma 2.1 ([5, Corollary 2.1]). Let R be a 2-torsion free semiprime ring, U a Lie ideal of R such that $U \nsubseteq Z(R)$ and $a, b \in U$.
(i) If $aUa = \{0\}$, then $a = 0$.
(ii) If $aU = \{0\}$ $(Ua = \{0\})$, then $a = 0$.
(iii) If U is a square closed Lie ideal and $aUb = \{0\}$, then $ab = 0$ and $ba = 0$.

Lemma 2.2 ([1, Theorem 3]). Let R be prime ring with $charR \neq 2$ and U be a nonzero Lie ideal of R. Let $B : R \times R \to R$ be a symmetric bi-derivation and f be the trace of B such that
(i) $f(U) = 0$, then $U \subseteq Z(R)$ or $f = 0$.
(ii) $f(U) \subseteq Z(R)$ and U be a square closed Lie ideal, then $U \subseteq Z(R)$ or $f = 0$.

Lemma 2.3 ([4, Lemma 1]). Let R be a 2-torsion free semiprime ring and U be a Lie ideal of R. Suppose that $[U,U] \subseteq Z(R)$, then $U \subseteq Z(R)$.

Lemma 2.4 ([2, Lemma 4]). Let R be a prime ring of characteristic different from 2 and $U \nsubseteq Z(R)$ be a Lie ideal of R and $a, b \in R$, if $aUb = \{0\}$ then $a = 0$ or $b = 0$.

3 Results on Prime ring

We start this section with the following lemma:

Lemma 3.1. Let R be a prime ring with $charR \neq 2$ and U be a square closed Lie ideal of R. Suppose that $B : R \times R \to R$ is a symmetric bi-derivation and f the trace of B such that $[f(x), y] \in Z(R)$ for all $x, y \in U$, then either $U \subseteq Z(R)$ or $f = 0$.

Proof. Suppose on the contrary that $U \nsubseteq Z(R)$. Since we have given that $[f(x), y] \in Z(R)$ for all $x, y \in U$. Replacing y by $2zy$ and using the fact that $charR \neq 2$, we get $[f(x), y]z + y[f(x), z] \in Z(R)$ for all $x, y, z \in U$. This implies that $[[f(x), y]z + y[f(x), z], r] = 0$ for all $x, y, z \in U$ and $r \in R$ i.e., $[f(x), y][z, r] + [y, r][f(x), z] = 0$ for all $x, y, z \in U$ and $r \in R$. Now, in particular Replacing r by z, we obtain $[y, z][f(x), z] = 0$ for all $x, y, z \in U$. Again, replacing y by $2yt$ and using $charR \neq 2$, we get $[y, z]t[f(x), z] = 0$ for all $x, y, z, t \in U$ i.e., $[y, z]U[f(x), z] = \{0\}$ for all $x, y, z \in U$. Thus in view of Lemma 2.4 we find that for each pair of $x, y, z \in U$ either $[y, z] = 0$ or $[f(x), z] = 0$. For each $z \in U$, let $A' = \{y \in U | [y, z] = 0\}$ and $B' = \{x \in U | [f(x), z] = 0\}$. Hence A' and B' are the additive subgroups of U whose union is U. By Brauer's trick, we have either $U = A'$ or $U = B'$. If $A' = U$, then $[y, z] = 0$ for all $y, z \in U$ and have $U \subseteq Z(R)$ a contradiction. On the other hand if $U = B'$, then $[f(x), z] = 0$ for all $x, z \in U$ and hence $f(U) \subseteq C_R(U) = Z(R)$, then by Lemma 2.2, we get $f = 0$. This completes the proof of the lemma. □

Theorem 3.1. Let R be a prime ring with $charR \neq 2$ and U be a square closed Lie ideal of R. Suppose that $B : R \times R \to R$ is a symmetric bi-derivation and f the trace of B. If $[f(x), x] = 0$ for all $x \in U$, then either $U \subseteq Z(R)$ or $f = 0$.

Proof. Suppose on the contrary that $U \nsubseteq Z(R)$. We have $[f(x), x] = 0$ for all $x \in U$. Replacing x by $x + y$ in the above expressions, we obtain $[f(x + y), x + y] = 0$ for all $x, y \in U$. This implies that $[f(x), y] + [f(y), x] + 2[B(x, y), x] + 2[B(x, y), y] = 0$ for all $x, y \in U$. Replacing x by $-x$ in the above expression, we get $[f(x), y] - [f(y), x] + 2[B(x, y), x] - 2[B(x, y), y] = 0$ for all $x, y \in U$. Combining above expressions and by $charR \neq 2$, we find that $[f(x), y] + 2[B(x, y), x] = 0$ for all $x, y \in U$. Replacing y by $2yz$ in the above expression, $2[f(x), y]z + 2y[f(x), z] + 4[B(x, yz), x] = 0$ for all $x, y, z \in U$. This gives $2B(x, y)[z, x] + 2[y, x]B(x, z) = 0$. In particular, $z = x$ we get $2[y, x]B(x, x) = 0$ for all $x, y \in U$. By $charR \neq 2$, we get $[x, y]B(x, x) = 0$ for all $x, y \in U$. Replacing y by $2yz$ and using the fact that $charR \neq 2$, we get $[x, y]zB(x, x) = 0$ for all $x, y, z \in U$. This gives $[x, y]UB(x, x) = 0$, by Lemma 2.4, for each $x \in U$ either $[x, y] = 0$ or $B(x, x) = 0$. In the first case it follows that by Lemma 2.3 that $x \in Z(R)$ for all $x \in U$. Thus if $x \notin Z(R)$, then $B(x, x) = 0$. Let $x, z \in U$ such that $x \in Z(R)$ and $z \notin Z(R)$. Hence $x + z \notin Z(R)$ and $x - z \notin Z(R)$. Thus $B(x + z, x + z) = 0$ and $B(x - z, x - z) = 0$. Adding the above two relations, we find that $2B(x, x) = 0$. Since $charR \neq 2$, we get $B(x, x) = 0$. Thus for all $x \in U$, $B(x, x) = 0$ and from Lemma 2.2 (i), $f = 0$. □

Theorem 3.2. Let R be a prime ring with $charR \neq 2$ and U be a square closed Lie ideal of R. Suppose that $B : R \times R \to R$ is a symmetric bi-derivation and f is the trace of B such that $f([x, y]) - [f(x), y] \in Z(R)$ for all $x, y \in U$. Then either $U \subseteq Z(R)$ or $f = 0$.

Proof. Suppose on the contrary that $U \nsubseteq Z(R)$. We have $f([x, y]) - [f(x), y] \in Z(R)$ for all $x, y \in U$. Replacing y by $y + z$ in the above expression, we obtain that $f([x, y+z]) - [f(x), y + z] \in Z(R)$ for all $x, y, z \in U$. This implies that $f([x, y]) + f([x, z]) + 2B([x, y], [x, z]) - [f(x), y] - [f(x), z] \in Z(R)$ for all $x, y, z \in U$. Now, using our hypothesis and $charR \neq 2$, we get $B([x, y], [x, z]) \in Z(R)$ for all $x, y, z \in U$. In particular, putting $z = y$, we find that $B([x, y], [x, y]) \in Z(R)$ for all $x, y \in U$ i.e., $f([x, y]) \in Z(R)$ for all $x, y \in U$. Combining the last expression with our hypothesis, we find that $[f(x), y] \in Z(R)$ for all $x, y \in U$. Thus, by Lemma 3.1, we get the required result. □

Theorem 3.3. Let R be a prime ring with $charR \neq 2$ and U a square closed Lie ideal of R. Suppose that $B : R \times R \to R$ is a symmetric bi-derivation and f the trace of B such that $f(x \circ y) - [f(x), y] \in Z(R)$ for all $x, y \in U$. Then either $U \subseteq Z(R)$ or $f = 0$.

Proof. Suppose on the contrary that $U \nsubseteq Z(R)$. We have $f(x \circ y) - [f(x), y] \in Z(R)$ for all $x, y \in U$. Replacing y by $y + z$ in the above expression, we obtain that $f(x \circ (y + z)) - [f(x), (y + z)] \in Z(R)$ for all $x, y, z \in U$. This implies that $f(x \circ y) + f(x \circ z) + 2B(x \circ y, x \circ z) - [f(x), y] - [f(x), z] \in Z(R)$ for all $x, y, z \in U$. Now, using our hypothesis and $charR \neq 2$, we get $B(x \circ y, x \circ z) \in Z(R)$ for all $x, y, z \in U$. In particular, putting $z = y$, we find that $B(x \circ y, x \circ y) \in Z(R)$ for all $x, y \in U$ i.e., $f(x \circ y) \in Z(R)$ for all $x, y \in U$. Combining the last step with our hypothesis, we find that $[f(x), y] \in Z(R)$ for all $x, y \in U$. Thus, by Lemma 3.1, we get $f = 0$. □

Theorem 3.4. Let R be a prime ring with $charR \neq 2$ and U a square closed Lie ideal of R. Suppose that $B : R \times R \to R$ is a symmetric bi-derivation and f is the trace of B such that $f(x) \circ y - [f(x), y] \in Z(R)$ for all $x, y \in U$. Then either $U \subseteq Z(R)$ or $f = 0$.

Proof. Suppose on contrary that $U \nsubseteq Z(R)$. Given that $f(x) \circ y - [f(x), y] \in Z(R)$ for all $x, y \in U$. This implies that $2yf(x) \in Z(R)$ for all $x, y \in U$, $charR \neq 2$ implies that $yf(x) \in Z(R)$ for all $x, y \in U$. Hence $[yf(x), r] = 0$ for all $x, y \in U$ and $r \in R$ i.e.,

$$y[f(x), r] + [y, r]f(x) = 0 \text{ for all } x, y \in U \text{ and } r \in R. \tag{3.1}$$

Replacing y by $2ty$ and using $charR \neq 2$, we obtain $t\{y[f(x), r] + [y, r]f(x)\} + [t, r]yf(x) = 0$ for all $x, y, t \in U$ and $r \in R$. Using (3.1), we get $[t, r]yf(x) = 0$ for all $x, y, t \in U$ and $r \in R$. This implies that $[t, r]Uf(x) = 0$ for all $x, t \in U$ and $r \in R$. By Lemma 2.4, we get either $[t, r] = 0$ or $f(x) = 0$ for all $x, t \in U$ and $r \in R$. If $[t, r] = 0$, then $U \subseteq Z(R)$ a contradiction. Hence if $f(x) = 0$ for all $x \in U$, then by Lemma 2.2 (i), we get $f = 0$. □

Theorem 3.5. Let R be a prime ring with $charR \neq 2$ and U be a square closed Lie ideal of R. Suppose that $B : R \times R \to R$ is a symmetric bi-derivation and f is the trace of B and $g : R \to R$ is any mapping such that $[f(x), y] - [x, g(y)] \in Z(R)$ for all $x, y \in U$. Then either $U \subseteq Z(R)$ or $f = 0$.

Proof. Suppose on the contrary that $U \nsubseteq Z(R)$. Since $[f(x), y] - [x, g(y)] \in Z(R)$ for all $x, y \in U$. Replacing x by $x+z$ in the above expression, we obtain that $[f(x+z), y] - [x+z, g(y)] \in Z(R)$ for all $x, y, z \in U$. This implies that $[f(x), y] + [f(z), y] + 2[B(x, z), y] - [x, g(y)] - [z, g(y)] \in Z(R)$ for all $x, y, z \in U$. Now, using our hypothesis and $charR \neq 2$, we get $[B(x, z), y] \in Z(R)$ for all $x, y, z \in U$. In particular, putting $z = x$, we find that $[B(x, x), y] \in Z(R)$ for all $x, y \in U$ i.e., $[f(x), y] \in Z(R)$ for all $x, y \in U$. Hence by Lemma 3.1, we get the required result. □

Theorem 3.6. Let R be a prime ring with $charR \neq 2$ and U be a square closed Lie ideal of R. Suppose that $B : R \times R \to R$ is a symmetric bi-derivation and f is the trace of B such that $f(x) \circ f(y) - [f(x), y] \in Z(R)$ for all $x, y \in U$. Then $U \subseteq Z(R)$ or $f = 0$.

Proof. Suppose on the contrary that $U \nsubseteq Z(R)$. We have $f(x) \circ f(y) - [f(x), y] \in Z(R)$ for all $x, y \in U$. Replacing y by $y+z$ in the above expression, we obtain that $f(x) \circ f(y) + f(x) \circ f(z) + 2f(x) \circ B(y, z) - [f(x), y] - [f(x), z] \in Z(R)$ for all $x, y, z \in U$. Now, using our hypothesis and $charR \neq 2$, we find that $f(x) \circ B(y, z) \in Z(R)$ for all $x, y, z \in U$. In particular, putting $z = y$, we get $f(x) \circ f(y) \in Z(R)$ for all $x, y \in U$. Combining the last step with our hypothesis, we find that $[f(x), y] \in Z(R)$ for all $x, y \in U$. Thus, by Lemma 3.1, we get the required result. □

Theorem 3.7. Let R be a prime ring with $charR \neq 2$ and U be a square closed Lie ideal of R. Suppose that $B : R \times R \to R$ is a symmetric bi-derivation and f is the trace of B and $g : R \to R$ be any mapping such that $f(x)y - xg(y) \in Z(R)$ for all $x, y \in U$. Then either $U \subseteq Z(R)$ or $f = 0$.

Proof. Suppose on the contrary that $U \nsubseteq Z(R)$. We have $f(x)y - xg(y) \in Z(R)$ for all $x, y \in U$. Replacing x by $x + z$ in the above expression, we obtain $f(x+z)y - (x+z)g(y) \in Z(R)$ for all $x, y, z \in U$. This implies that $f(x)y + f(z)y + 2B(x,z)y - xg(y) - zg(y) \in Z(R)$ for all $x, y, z \in U$. Using our hypothesis and $charR \neq 2$, we find that $B(x,z)y \in Z(R)$ for all $x, y, z \in U$. In particular $z = x$, we get $B(x,x)y \in Z(R)$ for all $x, y \in U$ i.e., $f(x)y \in Z(R)$ for all $x, y \in U$. This implies that $[f(x)y, r] = 0$ for all $x, y \in U$ and $r \in R$ i.e., $f(x)[y,r] + [f(x), r]y = 0$ for all $x, y \in U$ and $r \in R$. Replacing y by $2yt$ and using the fact that $charR \neq 2$, we get $f(x)y[t,r] + \{f(x)[y,r] + [f(x),r]y\}t = 0$ for all $x, y, t \in U$ and $r \in R$. Therefore we obtain, $f(x)y[t,r] = 0$ for all $x, y, t \in U$ and $r \in R$. Hence using the same arguments as used in the last paragraph of proof of Theorem 3.4, we get the required result. □

Theorem 3.8. Let R be a prime ring with $charR \neq 2$ and U is a square closed Lie ideal of R. Suppose that $B : R \times R \to R$ is a symmetric bi-derivation and f the trace of B such that $f(xy) - f(x)y - xf(y) \in Z(R)$ holds for all $x, y \in U$. Then either $U \subseteq Z(R)$ or $f = 0$.

Proof. Suppose on the contrary that $U \nsubseteq Z(R)$. Given that $f(xy) - f(x)y - xf(y) \in Z(R)$ holds for all $x, y \in U$. Replacing x by $x + z$ in the above relation, we obtain $f(xy) + f(zy) + 2B(xy, zy) - f(x)y - f(z)y - 2B(x,z)y - xf(y) - zf(y) \in Z(R)$ for all $x, y, z \in U$. Using our hypothesis, we conclude that $2B(xy, zy) - 2B(x,z)y \in Z(R)$ for all $x, y, z \in U$. Since $charR \neq 2$, then $B(xy, zy) - B(x,z)y \in Z(R)$ for all $x, y, z \in U$. In particular, putting $z = x$, we get

$$f(xy) - f(x)y \in Z(R) \text{ for all } x, y \in U. \tag{3.2}$$

Replacing y by $y + z$ in (3.2), we get $f(xy) + f(xz) + 2B(xy, xz) - f(x)y - f(x)z \in Z(R)$ for all $x, y, z \in U$. Now, using relation (3.2), we arrive at $2B(xy, xz) \in Z(R)$ for all $x, y, z \in U$. Again, since $charR \neq 2$, we get $B(xy, xz) \in Z(R)$ for all $x, y, z \in U$. In particular $z = y$, we get $f(xy) \in Z(R)$ for all $x, y \in U$. Again using relation (3.2), we have $f(x)y \in Z(R)$ for all $x, y \in U$. This means that $[f(x)y, r] = 0$ for all $x, y \in U$ and $r \in R$. This can be re-written as $f(x)[y,r] + [f(x), r]y = 0$ for all $x, y \in U$ and $r \in R$. In particular, putting $r = f(x)$, we get $f(x)[f(x), y] = 0$ for all $x, y \in U$ and $r \in R$. Replacing y by $2yz$ and using that $charR \neq 2$, we conclude that

$$f(x)y[f(x), z] = 0 \text{ for all } x, y, z \in U. \tag{3.3}$$

Multiplying the above equation left by z, we get $zf(x)y[f(x), z] = 0$ for all $x, y, z \in U$. Replacing y by $2zy$ in relation (3.3) and using the fact that $charR \neq 2$, we get $f(x)zy[f(x), z] = 0$ for all $x, y, z \in U$. Now combining the last two expressions, we find that $[f(x), z]y[f(x), z] = 0$ for all $x, y, z \in U$ that is $[f(x), z]U[f(x), z] = \{0\}$. Using Lemma 2.1, we get $[f(x), z] = 0$ for all $x, z \in U$ and hence by Lemma 3.1, we get $f = 0$. □

Theorem 3.9. Let R be a prime ring with $charR \neq 2$ and U be a square closed Lie ideal of R. Suppose that $B : R \times R \to R$ is a symmetric bi-derivation and f the trace of B such that $f(xy) - yf(x) - f(y)x \in Z(R)$ holds for all $x, y \in U$. Then either $U \subseteq Z(R)$ or $f = 0$.

Proof. Suppose on the contrary that $U \nsubseteq Z(R)$. Given that $f(xy) - yf(x) - f(y)x \in Z(R)$ holds for all $x, y \in U$. Replacing x by $x + z$ in the above relation, we obtain $f(xy) + f(zy) + 2B(xy, zy) - yf(x) - yf(z) - 2yB(x,z) - f(y)x - f(y)z \in Z(R)$ for all $x, y, z \in U$. Then using our hypothesis and $charR \neq 2$, we get $B(xy, zy) - yB(x,z) \in Z(R)$ for all $x, y, z \in U$. In particular, putting $z = x$, we find that $f(xy) - yf(x) \in Z(R)$ holds for all $x, y \in U$. Combining this with our hypothesis, we obtain $f(y)x \in Z(R)$ for all $x, y \in U$. This gives $[f(y)x, r] = 0$ for all $x, y \in U$ and $r \in R$. This yields that

$$f(y)[x,r] + [f(y), r]x = 0 \text{ holds for all } x, y \in U \text{ and } r \in R. \tag{3.4}$$

Replacing x by $2xz$ and using $charR \neq 2$, we find that $\{f(y)[x,r] + [f(y),r]x\}z + f(y)x[z,r] = 0$ holds for all $x, y, z \in U$ and $r \in R$. Using relation (3.4), we get $f(y)x[z,r] = 0$ for all $x, y, z \in U$ and $r \in R$. Using the same technique as we have used in Theorem 3.4, we get the result. □

Theorem 3.10. Let R be a prime ring with $charR \neq 2$ and U be a square closed Lie ideal of R. Suppose that $B : R \times R \to R$ is a symmetric bi-derivation and f the trace of B such that $f(xy) - xf(y) - yf(x) \in Z(R)$ holds for all $x, y \in U$. Then either $U \subseteq Z(R)$ or $f = 0$.

Proof. Suppose on the contrary that $U \nsubseteq Z(R)$. Given that $f(xy)-xf(y)-yf(x) \in Z(R)$ holds for all $x, y \in U$. Replacing x by $x+z$ in the above relation, we get $f(xy)+f(zy)+2B(xy, zy)-xf(y) - zf(y) - yf(x) - yf(z) - 2yB(x, z) \in Z(R)$ for all $x, y, z \in U$. Combining this with our hypothesis, we obtain $2B(xy, zy) - 2yB(x, z) \in Z(R)$ for all $x, y, z \in U$. $charR \neq 2$ yields that $B(xy, zy) - yB(x, z) \in Z(R)$ holds for all $x, y, z \in U$. In particular, putting $z = x$, we get $f(xy) - yf(x) \in Z(R)$ for all $x, y \in U$. Using the last expression with our hypothesis, we find that $xf(y) \in Z(R)$ holds for all $x, y \in U$. This gives that $[xf(y), r] = 0$ holds for all $x, y \in U$ and $r \in R$. Now, using the similar argument as used in the last paragraph of the proof of Theorem 3.4, we get required result. □

Theorem 3.11. Let R be a prime ring with $charR \neq 2$ and U be a square closed Lie ideal of R. Suppose that $B : R \times R \to R$ is a symmetric bi-derivation and f the trace of B such that $f([x, y]) - [f(x), y] - [x, f(y)] \in Z(R)$ holds for all $x, y \in U$. Then either $U \subseteq Z(R)$ or $f = 0$.

Proof. Suppose on the contrary that $U \nsubseteq Z(R)$. We have given that $f([x, y]) - [f(x), y] - [x, f(y)] \in Z(R)$ holds for all $x, y \in U$. Replacing x by $x + z$ in the above relation, we find that $f([x, y]) + f([x, y]) + 2B([x, y], [z, y]) - [f(x), y] - [f(z), y] - 2[B(x, y), y] - [x, f(y)] - [z, f(y)] \in Z(R)$ for all $x, y, z \in U$. Combining our hypothesis with above relation, we get $2B([x, y], [z, y]) - 2[B(x, y), y] \in Z(R)$ for all $x, y, z \in U$. Since $charR \neq 2$, we obtain $B([x, y], [z, y]) - [B(x, y), y] \in Z(R)$ for all $x, y, z \in U$. In particular, putting $z = x$, we find that

$$f([x, y]) - [f(x), y] \in Z(R) \text{ for all } x, y \in U. \tag{3.5}$$

Again replacing y by $y+z$ in the above relation, we arrive at $f([x, y])+f([x, z])+2B([x, y], [x, z])-[f(x), y]-[f(x), z] \in Z(R)$ for all $x, y, z \in U$. Using the relation (3.5) in the last expression, we get $2B([x, y], [x, z]) \in Z(R)$ for all $x, y, z \in U$. Since $charR \neq 2$, we have $B([x, y], [x, z]) \in Z(R)$ for all $x, y, z \in U$. In particular putting $z = y$, we get $f([x, y]) \in Z(R)$ for all $x, y \in U$. Now, combining the above relation with (3.5), we find that $[f(x), y] \in Z(R)$ for all $x, y \in U$. Using Lemma 3.1, we get the required result. □

4 Results on Semiprime ring

Theorem 4.1. Let R be a 2-torsion free semiprime ring and U be a Lie ideal of R. Suppose that $A : R \times R \to R$ is a symmetric bi-additive mapping and f is the trace of A such that $f([x, y]) - [x, y] \in Z(R)$ for all $x, y \in U$. Then $U \subseteq Z(R)$.

Proof. We have $f([x, y]) - [x, y] \in Z(R)$ for all $x, y \in U$. Replacing x by $x + z$ in the above expression, we obtain that $f([x, y]) + f([z, y]) + 2A([x, y], [z, y]) - [x, y] - [z, y] \in Z(R)$ for all $x, y, z \in U$. Now, using our hypothesis and the fact that R is 2-torsion free, we get $A([x, y], [z, y]) \in Z(R)$ for all $x, y, z \in U$. In particular, putting $z = x$, we find that $A([x, y], [x, y]) \in Z(R)$ for all $x, y \in U$ i.e., $f([x, y]) \in Z(R)$. Combining the last step with our hypothesis, we find that $[x, y] \in Z(R)$ for all $x, y \in U$ i.e., $[U, U] \in Z(R)$. Then by Lemma 2.3, we get the required result. □

Theorem 4.2. Let R be a 2-torsion free semiprime ring and U be a square closed Lie ideal of R. Suppose that $A : R \times R \to R$ is a symmetric bi-additive mapping and f is the trace of A such that $f(x \circ y) - (x \circ y) \in Z(R)$ for all $x, y \in U$. Then $U \subseteq Z(R)$.

Proof. Suppose on the contrary that $U \nsubseteq Z(R)$. We have $f(x\circ y)-x\circ y \in Z(R)$ for all $x, y \in U$. Replacing x by $x + z$ in above expression, we obtain that, $f(x \circ y) + f(z \circ y) + 2A(x \circ y, z \circ y) - x \circ y - z \circ y \in Z(R)$ for all $x, y, z \in U$. Now, using our hypothesis and the fact that R is 2-torsion free, we get $A(x \circ y, z \circ y) \in Z(R)$ for all $x, y, z \in U$. In particular, putting $z = x$, we find that $A(x \circ y, x \circ y) \in Z(R)$ for all $x, y \in U$ i.e., $f(x \circ y) \in Z(R)$. Combining the last step with our hypothesis, we find that $x \circ y \in Z(R)$ for all $x, y \in U$. Replacing x by $2yx$, we get $2y(x \circ y) \in Z(R)$ for all $x, y \in U$. This implies that $[2y(x \circ y), z] = 0$ for all $x, y, z \in U$. On solving and using the fact that R is 2-torsion free, we conclude that $[y, z](x \circ y) = 0$ for all $x, y, z \in U$. Again replacing x by $2xz$ and using the fact that R is 2-torsion free, we get $[y, z]x[z, y] = 0$ for all $x, y, z \in U$. By Lemma 2.1, we get $U \subseteq Z(R)$, a contradiction. □

Theorem 4.3. Let R be a 2-torsion free semiprime ring and U be a square closed Lie ideal of R. Suppose that $A : R \times R \to R$ is a symmetric bi-additive mapping and f is the trace of A such that $f([x,y]) - (x \circ y) \in Z(R)$ for all $x, y \in U$. Then $U \subseteq Z(R)$.

Proof. Suppose on the contrary that $U \nsubseteq Z(R)$. Given that $f([x,y]) - (x \circ y) \in Z(R)$ for all $x, y \in U$. Replacing x by $x+z$ in the above expression, we obtain that $f([x+z,y])-(x+z)\circ y \in Z(R)$ for all $x, y, z \in U$. This implies that $f([x,y])+f([z,y])+2A([x,y],[z,y])-[x,y]-[z,y] \in Z(R)$ for all $x, y, z \in U$. Now, using our hypothesis and the fact that R is 2-torsion free, we get $A([x,y],[z,y]) \in Z(R)$ for all $x, y, z \in U$. In particular, putting $z = x$, we find that $A([x,y],[x,y]) \in Z(R)$ for all $x, y \in U$ i.e., $f([x,y]) \in Z(R)$. Combining the last step with our hypothesis, we find that $x \circ y \in Z(R)$ for all $x, y \in U$. Now, the same steps as we have used in Theorem 4.2 we get the required result. □

Theorem 4.4. Let R be a 2-torsion free semiprime ring and U be a Lie ideal of R. Suppose that $A : R \times R \to R$ is a symmetric bi-additive mapping and f is the trace of A such that $f(x \circ y) - [x,y] \in Z(R)$ for all $x, y \in U$. Then $U \subseteq Z(R)$.

Proof. We have given that $f(x \circ y) - [x,y] \in Z(R)$ for all $x, y \in U$. Replacing x by $x + z$ in the above expression, we obtain that $f((x+z)\circ y) - [x+z,y] \in Z(R)$ for all $x, y, z \in U$. This implies that $f(x \circ y) + f(z \circ y) + 2A(x \circ y, z \circ y) - [x,y] - [z,y] \in Z(R)$ for all $x, y, z \in U$. Now, using our hypothesis and the fact that R is 2-torsion free, we get $A(x \circ y, z \circ y) \in Z(R)$ for all $x, y, z \in U$. In particular, putting $z = x$, we find that $A(x \circ y, x \circ y) \in Z(R)$ for all $x, y \in U$ i.e., $f(x \circ y) \in Z(R)$. Combining the last step with our hypothesis, we find that $[x,y] \in Z(R)$ for all $x, y \in U$ i.e., $[U,U] \subseteq Z(R)$. Then, by Lemma 2.3, we get the required result. □

Theorem 4.5. Let R be a 2-torsion free semiprime ring and U be a square closed Lie ideal of R. Suppose that $A : R \times R \to R$ is a symmetric bi-additive mapping and f the trace of A such that $2(x \circ y) = f(x) - f(y)$ for all $x, y \in U$. Then $U \subseteq Z(R)$.

Proof. Suppose on the contrary that $U \nsubseteq Z(R)$. Since we have $2(x \circ y) = f(x) - f(y)$ for all $x, y \in U$. Replacing x by $x + y$ in the above expression, we obtain $4y^2 = 2A(x,y) + f(y)$ for all $x, y \in U$. Replacing x by $-x$ in above expression, we get $4y^2 = -2A(x,y) + f(y)$ for all $x, y \in U$. Now, combining the last two expression, we obtain $4y^2 = f(y)$ for all $x, y \in U$. Putting $y = x$ in our hypothesis, we find that $4y^2 = 0$. This implies that $f(y) = 0$ for all $y \in U$. Hence $2(x \circ y) = 0$ for all $x, y \in U$. Since R is 2-torsion free, we get $x \circ y = 0$ for all $x, y \in U$. Using the same argument as used in the proof of the Theorem 4.2, we get the required result. □

Theorem 4.6. Let R be a 2-torsion free semiprime ring and U be a square closed Lie ideal of R. Suppose that $A : R \times R \to R$ is a symmetric bi-additive mapping and f is the trace of A such that $f(x) \circ f(y) - x \circ y \in Z(R)$ for all $x, y \in U$. Then $U \subseteq Z(R)$.

Proof. Suppose on the contrary that $U \nsubseteq Z(R)$. We have $f(x) \circ f(y) - x \circ y \in Z(R)$ for all $x, y \in U$. Replacing y by $y + z$ in the above expression, we obtain that $f(x) \circ f(y) + f(x) \circ f(z) + 2f(x) \circ A(y,z) - x \circ y - x \circ z \in Z(R)$ for all $x, y, z \in U$. Now, using our assumption and the fact that R is 2-torsion free, we find that $f(x) \circ A(y,z) \in Z(R)$ for all $x, y, z \in U$. In particular, putting $z = y$, we get $f(x) \circ f(y) \in Z(R)$ for all $x, y \in U$. Combining the last step with our hypothesis, we find that $x \circ y \in Z(R)$ for all $x, y \in U$. Then using the similar technique as used in Theorem 4.2, we get the required result. □

Theorem 4.7. Let R be a 2-torsion free semiprime ring and U be a Lie ideal of R. Suppose that $A : R \times R \to R$ is a symmetric bi-additive mapping and f is the trace of A such that $f(x) \circ f(y) - [x,y] \in Z(R)$ for all $x, y \in U$. Then $U \subseteq Z(R)$.

Proof. Given that $f(x) \circ f(y) - [x,y] \in Z(R)$ for all $x, y \in U$. Replacing y by $y + z$ in the above expression, we obtain that $f(x) \circ f(y) + f(x) \circ f(z) + 2f(x) \circ A(y,z) - [x,y] - [x,z] \in Z(R)$ for all $x, y, z \in U$. Now, using our assumption and fact that R is 2-torsion free, we find that $f(x) \circ A(y,z) \in Z(R)$ for all $x, y, z \in U$. In particular, putting $z = y$, we get $f(x) \circ f(y) \in Z(R)$ for all $x, y \in U$. Combining the last step with our assumption, we find that $[x,y] \in Z(R)$ for all $x, y \in U$. Thus, by Lemma 2.3, we get the required result. □

Theorem 4.8. Let R be a 2-torsion free semiprime ring and U be a Lie ideal of R. Suppose that $A : R \times R \to R$ is a symmetric bi-additive mapping and f is the trace of A such that $xy - f(x) = yx - f(y)$ holds for all $x, y \in U$. Then $U \subseteq Z(R)$).

Proof. We have $xy - f(x) = yx - f(y)$ for all $x, y \in U$. This can be re-written as $[x, y] = f(x) - f(y)$ for all $x, y \in U$. Replacing x by $x + y$ in the above relation, we obtained, $[x, y] = f(x) - 2A(x, y)$ for all $x, y \in U$. Now, substituting $-x$ in place of x and combining the above relation, we get $2f(x) = 0$ for all $x, y \in U$. Since R is 2-torsion free, we find that $f(x) = 0$ for all $x \in U$. Now, combining it with our hypothesis, we arrive at $[x, y] = 0$ for all $x, y \in U$. Hence, by Lemma 2.3, we get $U \subseteq Z(R)$. □

Theorem 4.9. Let R be a 2-torsion free semiprime ring and U be a square closed Lie ideal of R. Suppose that $B : R \times R \to R$ is a symmetric bi-derivation and f the trace of B such that $[x, y] = f(xy) - f(yx)$ holds for all $x, y \in U$. Then $U \subseteq Z(R)$.

Proof. Given that $[x, y] = f(xy) - f(yx)$ holds for all $x, y \in U$. This can be re-written as

$$[x, y] = [x^2, f(y)] + [f(x), y^2] + 2xB(x, y)y - 2yB(x, y)x \text{ for all } x, y \in U. \tag{4.1}$$

Now, replacing x by $x + y$ in (4.1), we obtained

$$\begin{aligned} [x, y] &= [x^2, f(y)] + [xy, f(y)] + [yx, f(y)] + [f(x), y^2] + 2[B(x, y), y^2] \\ &+ 2xB(x, y)y + 2xf(y)y - 2yB(x, y)x - 2yf(y)x \text{ for all } x, y \in U. \end{aligned} \tag{4.2}$$

Thus in view of expression of (4.1) yields that

$$0 = [xy, f(y)] + [yx, f(y)] + 2[B(x, y), y^2] + 2xf(y)y - 2yf(y)x \text{ for all } x, y \in U. \tag{4.3}$$

Replacing x by $x + y$ in (4.2) and using (4.2), we obtained

$$2([x^2, f(y)] + [f(x), y^2] + 2xB(x, y)y - 2yB(x, y)x) = 0 \text{ for all } x, y \in U. \tag{4.4}$$

Since R is 2-torsion free, the last expression implies that $[x, y] = [x^2, f(y)] + [f(x), y^2] + 2xB(x, y)y - 2yB(x, y)x) = 0$ for all $x, y \in U$. This yields that $U \subseteq Z(R)$. □

Theorem 4.10. Let R be a 2-torsion free semiprime ring and U be a square closed Lie ideal of R. Suppose that $B : R \times R \to R$ is a symmetric bi-derivation and f is the trace of B such that $[x, y] - f(xy) + f(yx) \in Z(R)$ holds for all $x, y \in U$. Then $U \subseteq Z(R)$).

Proof. We have $[x, y] - f(xy) + f(yx) \in Z(R)$ for all $x, y \in U$. This can be re-written as

$$[x, y] - [x^2, f(y)] - [f(x), y^2] - 2xB(x, y)y + 2yB(x, y)x \in Z(R) \text{ for all } x, y \in U. \tag{4.5}$$

Now using the similar argument as we have used form (4.1) to (4.3), we get

$$0 = [xy, f(y)] + [yx, f(x)] + 2[B(x, y), y^2] + 2xf(y)y - 2yf(y)x \in Z(R) \text{ for all } x, y \in U. \tag{4.6}$$

Further replacing y by $x + y$ in the last expression and using the fact that R is 2-torsion free, we find that $f(xy) - f(yx) \in Z(R)$ for all $x, y \in U$. Combining this our hypothesis, we get $[x, y] \in Z(R)$ for all $x, y \in U$. Hence using Lemma 2.3, we get the required result. □

References

[1] N. Argac, and M. S. Yenigul, *Lie ideals and symmetric bi-derivation on prime and semiprime rings*, Pure and Applied Math. Sci. 44(1-2), 17-21 (1996).

[2] J. Bergen, I. N. Herstein and J.W. Kerr, *Lie ideals and derivations of prime rings*, J. Algebra 71, 259-267 (1981).

[3] M. Bresar, *Commuting maps: A survey*, Taiwanese J. Math. 8(3), 361-397 (2004).

[4] I. N. Herstein, *On the Lie structure of an associative ring*, Journal of Algebra 14, 561-571 (1970).

[5] M. Hongan, N. Rehman and R. M. Al Omary, *Lie ideals and Jordan triple derivations of rings*, 125, 147-156 (2011).

[6] G. Maksa, *A remark on symmetric bi-additive functions having non-negative diagonalization*, Glasnic Math. 15, 279-280 (1980).

[7] G. Maksa, *On trace of symmetric bi-derivation*, C. R. Math. Rep. Acad. Sci. Canada 9, 303-307 (1987)

[8] N. M. Muthana, *Left cetralizer traces, generalized bi-derivations left bimultipliers and generalized Jordan biderivations*, The aligarh Bull. of Maths. 26(2), 33-45 (2007).

[9] E. C. Posner, *Derivations in prime rings*, Proc. Amer. Math. Soc. 8, 1093-1100 (1957).

[10] J. Vukman, *symmetric bi-derivation on prime and semiprime rings*, Aequations Math. 38, 245-254 (1989).

[11] J. Vukman, *Two results concerning symmetric bi-derivation on prime rings*, Aequations Math. 40, 181-189 (1990).

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