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Hollow Modules Over Commutative Rings

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Dedicated to Patrick Smith and John Clark on the occasion of their 70th birthdays.

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Abstract. Let R be a commutative ring with identity. An R-module M is said to be hollow, if the set of maximal submodules of M is empty or for every maximal submodule N of M, and for every submodule K of M, the equality $N + K = M$, implies that $K = M$.

Some equivalent conditions for a module to be hollow are given. Hollow modules over Laskerian domains and over arithmetical rings are studied. Furthermore modules whose submodules are hollow will be characterized, and the behavior of hollow modules under the localization is studied.

1 Introduction

According to [9] a non-zero module M is define to be hollow if every submodule N of M is small, that is for any submodule K of M, the equality $N + K = M$ implies that $K = M$.

In noncommutative rings, the above classical notion of hollow modules has been studied extensively for a long time in many papers (see for example [7, 9, 10, 11, 12, 13, 14, 15, 16, 19, 25, 26]). Some interesting results on hollow modules over noncommutative rings can be found in [26, Chapter 8].

Throughout this paper all the rings considered are commutative with identity, all modules are unitary. The set of maximal submodules of a module M is denoted by $Max(M)$.

In this paper a slightly different notion of hollow modules is introduced and studied as follows:

Definition 1.1. An R-module M is said to be a hollow module, if $Max(M) = \emptyset$, or for every maximal submodule N of M, and for every submodule K of M, the equality $N + K = M$, implies that $K = M$.

Unlike the classical definition of hollow module, in this new notion of hollow modules, the module $M = 0$ is hollow. So whenever we need to proof a module M is hollow, we can assume that $0 \neq M$ and $Max(M) \neq \emptyset$.

Also according to this definition Z_{p^∞} is a hollow Z-module, since $Max(Z_{p^\infty}) = \emptyset$. Besides if R is an integral domain which is not a field and K is the field of fractions of R, then K is a hollow R-module, since $Max(K) = \emptyset$ (see [21, Theorem 1]).

Several characterizations of hollow modules are given, in Section 2 of this paper. We will find some equivalent conditions for a module to be hollow; we will study when every submodule of a module is a hollow module.

We will prove that an integral domain R of Krull dimension ≤ 1 is Laskerian if and only if every cyclic R-module is either isomorphic to R or isomorphic to a finite direct sum of hollow modules, and when this is the case, we show that the global dimension of R can be computed from only the dimensions of cyclic hollow modules.

In Section 4, we will find the relation between the localization of a module and hollowness. Particularly in 4.2, we will get a characterization for locally cyclic modules.

2 Equivalent conditions for hollowness

The main results in this section gives a list of equivalent conditions that characterize hollow modules. In this section, we will show that hollow modules are closely related to multiplication modules. Recall that an R-module M is said to be multiplication if $N = (N : M)M$ for every submodule N of M, where $(N : M) = \{r \in R \mid rM \subseteq N\}$ (see, [1, 5, 8]).

Also recall that an R-module M is locally cyclic, if for each prime ideal P of R, M_P is a cyclic R_P-module.

The following lemma will be used throughout this paper.

Lemma 2.1. *Let R be a ring. Then:*

(i) *[5, Proposition 4] Every multiplication module over a semi local ring is cyclic.*

(ii) *[1, Theorem 2.1(1),(6)] If M is a multiplication R-module, and N is a submodule of M, then for each prime ideal P of R, M_P is a cyclic R_P-module and if $M_P \neq 0$, then $(N : M)_P = (N_P : M_P)$.*

(iii) *[5, Lemma 2(ii)] If M is a finitely generated R-module and for each maximal ideal $\mathfrak{M}$ of R, $M_{\mathfrak{M}}$ is a multiplication $R_{\mathfrak{M}}$-module, then M is a multiplication R-module.*

(iv) *[8, Corollary 2.3] Let $M_\lambda,\ \lambda \in \Lambda$, be a collection of finitely generated R-modules, and $M = \oplus_{\lambda\in\Lambda} M_\lambda$. Then M is a multiplication module if and only if for each $\lambda \in \Lambda$, M_λ is a multiplication module and $Ann\ (M_\lambda) + Ann\ \hat{M}_\lambda = R$.*

(v) *[5, Lemma 3] If M is a locally cyclic R-module, and $\frac{R}{Ann\ M}$ is a semi local ring, then M is a cyclic R-module.*

Recall that a local module is a module with exactly one maximal submodule. Also a local ring is a ring (not necessarily Noetherian) with exactly one maximal ideal.

Theorem 2.2. *Let M be a non-zero R-module. Then the following are equivalent:*

(i) *M is a hollow module and $Max(M) \neq \emptyset$;*

(ii) *M is a cyclic and local module;*

(iii) *M is a finitely generated local module;*

(iv) *$\frac{R}{Ann\ M}$ is a local ring and M is a cyclic module;*

(v) *$\frac{R}{Ann\ M}$ is a local ring and M is a multiplication module.*

(vi) *There exists an ideal I of R where $M \cong \frac{R}{I}$ and $\frac{R}{I}$ is a local ring.*

Proof. **(i)** $\Longrightarrow$ **(ii)** Let N be a maximal submodule of M and let L be an arbitrary submodule of M where $L \not\subseteq N$. Since $N + L = M$, and M is a hollow module, then $M = L$. Hence M has just one maximal submodule. If $x \in M - N$ and $L = Rx$, then $M = Rx$.

(ii) $\Longrightarrow$ **(iii)** The proof is obvious.

(iii) $\Longrightarrow$ **(i)** Let N be a maximal submodule of M and let L be a submodule of M. If $N + L = M$ and $L \neq M$, then by Zorn's Lemma there exists a maximal submodule N_0 of M containing L. Since M is a local module, $N_0 = N$. So $L \subseteq N_0 = N$ and consequently $M = N + L = N$ which is a contradiction. Thus $L = M$.

(ii) $\Longrightarrow$ **(iv)** Since M is cyclic, so $M \cong \frac{R}{Ann\ M}$. Since M is a local R-module, $\frac{R}{Ann\ M}$ is a local R-module. It is easy to see that the submodules of $\frac{R}{Ann\ M}$ as an R-module are exactly the ideals of the ring $\frac{R}{Ann\ M}$. Hence $\frac{R}{Ann\ M}$ is a local ring.

(iv) $\Longrightarrow$ **(v)** The proof is clear.

(v) $\Longrightarrow$ **(ii)** Obviously M is a multiplication $\frac{R}{Ann\ M}$-module, and since $\frac{R}{Ann\ M}$ is a local ring, so by 2.1(**i**), M is a cyclic $\frac{R}{Ann\ M}$-module and clearly it is a cyclic R-module. Now $M \cong \frac{R}{Ann\ M}$ and hence M is a local R-module as well.

(iv) $\Longrightarrow$ **(vi)** Since $\frac{R}{Ann\ M}$ is a local ring and M is a cyclic R-module, $M \cong \frac{R}{Ann\ M}$. That is, in this case $I = Ann\ M$.

(vi) $\Longrightarrow$ **(iv)** Obviously $M \cong \frac{R}{I}$ is a cyclic R-module which is generated by $1 + I$. So M is a non-zero cyclic R-module. Also $Ann\ M = Ann\ _R(\frac{R}{I}) = I$. This means, $\frac{R}{Ann\ M} = \frac{R}{I}$ is a local ring. □

Corollary 2.3. *Let $0 \neq R$ be a ring. Then the following are equivalent:*

(i) *Every cyclic R-module is a hollow module;*

(ii) *R is a local ring.*

Proof. If every cyclic R-module is a hollow module, then obviously R as an R-module is hollow. So by 2.2(**ii**), R is a local R-module and thus R is a local ring.

Conversely let R be a local ring and let M be a non-zero cyclic R-module. Clearly $\frac{R}{Ann\ M}$ is a local ring. Hence by 2.2(**iv**), M is a hollow module. □

Corollary 2.4. *Let R be an integral domain of Krull dimension one, and let M be a non-zero multiplication R-module with $Ann\ M \neq 0$. Then M is a hollow module if and only if $Ann\ M$ is a primary ideal.*

Proof. By 2.2(**v**), M is a hollow module if and only if $\frac{R}{Ann\ M}$ is a local ring. Hence it is enough to show that $\frac{R}{Ann\ M}$ is a local ring if and only if $Ann\ M$ is a primary ideal. This is a consequence of the following well known facts:

If an ideal is contained in only one prime ideal P, then it is P-primary, and if an ideal I is m-primary where m is a maximal ideal, then m is the unique prime ideal containing I (see [6, Page 268, Remark]). □

3 Hollow modules over Laskerian domains

Recall that a ring R is called Laskerian if every ideal of R has a primary decomposition.

Theorem 3.1. *Let R be an integral domain of Krull dimension ≤ 1. Then R is a Laskerian domain if and only if every cyclic R-module is either isomorphic to R or isomorphic to a finite direct sum of hollow modules.*

Proof. If $dim\ R = 0$, then since R is an integral domain, it is a field, so the proof for the both sides of implication is obvious. Now suppose that $dim\ R = 1$.

($\Longrightarrow$) Let M be a cyclic R-module, which is not isomorphic to R. Then $M \cong \frac{R}{I}$, where I is a non-zero ideal of R. Since R is a Laskerian ring, then I has a minimal primary decomposition, say $I = \cap_{i=1}^{n} I_k$. Suppose $\sqrt{I_k} = P_k$, for $1 \leq k \leq n$. As $dim\ R = 1$, for each $k, k',\ k \neq k'$, the maximal ideals $P_k, P_{k'}$ are coprime and so $I_k, I_{k'}$ are coprime. Hence by [23, Theorems 1.4, 1.3], $\frac{R}{I} \cong \frac{R}{I_1} \oplus \frac{R}{I_2} \oplus \cdots \oplus \frac{R}{I_n}$ as rings, by the rule $\varphi(x + I) = (x + I_1, x + I_2, \cdots, x + I_n)$. One can easily see that φ is also an R-module isomorphism, that is $\frac{R}{I} \cong \frac{R}{I_1} \oplus \frac{R}{I_2} \oplus \cdots \oplus \frac{R}{I_n}$ as R-modules.

Note that for each k, the annihilator of the R-module $\frac{R}{I_k}$ is I_k, and it is a non-zero primary ideal. Thus by 2.4, $\frac{R}{I_k}$ is a hollow R-module, which completes the proof.

($\Longleftarrow$) Let I be a non-zero ideal of R. By our assumption, $\frac{R}{I} \cong \frac{R}{I_1} \oplus \frac{R}{I_2} \oplus \cdots \oplus \frac{R}{I_n}$, where each $\frac{R}{I_k}$ is a hollow R-module. Therefore $I = Ann\ \frac{R}{I} = Ann\ (\frac{R}{I_1} \oplus \frac{R}{I_2} \oplus \cdots \oplus \frac{R}{I_n}) = \cap_{k=1}^{n} I_k$. Since each $\frac{R}{I_k}$ is a hollow R-module, by 2.4, each I_k is a primary ideal, which completes the proof. □

Corollary 3.2. *If R is a Dedekind domain, then every finitely generated R-module is a finite direct sum of hollow modules and a finitely generated projective module.*

Proof. Let M be a finitely generated R-module. As R is a Dedekind domain, M is isomorphic to a direct sum of a finite number of cyclic modules and a projective module. Suppose $M \cong M_1 \oplus M_2 \oplus \cdots \oplus M_n \oplus P$, where each M_i is a cyclic module and P is a projective module. Note that $P \cong \frac{M}{M_1 \oplus M_2 \oplus \cdots \oplus M_n}$, so P is a finitely generated module.

Without loss of generality, we may suppose that $M_i \ncong R$ for $1 \leq i \leq k$, and $M_i \cong R$ for $k + 1 \leq i \leq n$ where k is a non-negative integer with $k \leq n$.

By 3.1, each M_i is a finite direct sum of hollow modules for each $1 \leq i \leq k$. Also note that $P' = M_{k+1} \oplus M_{k+2} \oplus \cdots \oplus M_n \oplus P \cong R \oplus R \oplus \cdots \oplus R \oplus P$, so P' is a finitely generated projective module. □

Recall that the projective dimension of a module M (see [24, Page 454]) is denoted by $pd_R(M)$ and the global dimension of a ring R denoted by $D(R)$ is defined by:

$$D(R) = \sup\{pd_R(M) :\ M \text{ is an } R-\text{module}\}.$$

According to [24, Theorem 8.16] a formula due to Auslander for simplifying the computation of $D(R)$ was given as follows:

$$D(R) = \sup\{pd_R(\frac{R}{I}) :\ I \text{ is an ideal of } R\}.$$

The flat dimension of a module M is denoted by $fd_R(M)$ and the weak global dimension of a ring R (see [24, Page 462]) denoted by $WD(R)$ is defined by:

$$WD(R) = \sup\{fd_R(M) :\ M \text{ is an } R-\text{module}\}.$$

According to [24, Theorem 8.25],

$$WD(R) = \sup\{fd_R(\frac{R}{I}) : \text{ } I \text{ is a non-zero ideal of } R\}.$$

Theorem 3.1 and the facts above show that the global dimensions of Laskerian domains of Krull dimension ≤ 1 can be computed from only the dimensions of cyclic hollow modules. The proof of the following result is evident and it is left to the reader.

Corollary 3.3. *Let R be a Laskerian domain of Krull dimension ≤ 1. Then*

(i) $D(R) = \sup\{pd_R(\frac{R}{I}) : I$ *is a non-zero primary ideal of* $R\}$
$= \sup\{pd_R(M) : M$ *is a non-zero hollow* $R\}$.

(ii) $WD(R) = \sup\{fd_R(\frac{R}{I}) : I$ *is a non-zero primary ideal of* $R\}$
$= \sup\{fd_R(M) : M$ *is a non-zero hollow* $R\}$.

In particular, R is a field if and only if every hollow R-module is flat.

4 Localization and Modules with Hollow Submodules

Recall that a proper submodule N of an R-module M is said to be a prime submodule of M, if the condition $ra \in N$, $r \in R$ and $a \in M$ implies that either $a \in N$ or $rM \subseteq N$. In this case, if $P = (N : M)$, then it is easy to see that P is a prime ideal of R, and we say that N is a P-prime submodule of M (see for example, [2], [4], [20], [22]).

Let N be a proper submodule of M and let $(N : M) = P$ be a prime ideal of R. It is easy to see that N is a prime submodule of M if and only if $\frac{M}{N}$ is a torsion-free $\frac{R}{P}$-module (see [20, Theorem 1]).

In this section we will investigate the relation between hollowness and localization. Also we will characterize module M such that every submodule of M is a hollow module.

Lemma 4.1. *[21, Proposition 1] Let M be an R-module, and S be a multiplicatively closed subset of R.*

(i) *If N is a P-prime submodule of M such that $P \cap S = \emptyset$, then $S^{-1}N$ is an $S^{-1}P$-prime submodule of $S^{-1}M$ as an $S^{-1}R$-module, and $(S^{-1}N) \cap M = N$.*

(ii) *If T is a Q-prime submodule of $S^{-1}M$ as an $S^{-1}R$-module, then $T \cap M$ is a $Q \cap R$-prime submodule of M, $S^{-1}(T \cap M) = T$, and $Q \cap S = \emptyset$.*

Let M be an R-module and P a prime ideal of R. We say that $dim_P\ M = 0$ if $N_1 = N_2$ for every two P-prime submodules N_1 and N_2 of M with $N_1 \subseteq N_2$ (see [2]).

Definition. An R-module M is called semi multiplication if for every prime ideal P of R, the R_P-module M_P is a finitely generated and $dim_P\ M = 0$.

Theorem 4.2. *Let M be an R-module. Then the following are equivalent:*

(i) *M is a semi multiplication R-module;*

(ii) *for any maximal ideal P of R, M_P is a finitely generated R_P-module with $dim_P\ M = 0$;*

(iii) *M is a locally cyclic module.*

Proof. **(i) $\Longrightarrow$ (ii)** The proof is clear.

(ii) $\Longrightarrow$ (iii) Let P be a prime ideal of R. We consider the following two cases:

Case 1. The ideal P is a maximal ideal of R.

Consider the vector space $\frac{M_P}{P_P M_P}$ over the field $\frac{R_P}{P_P}$. We show that $rank_{\frac{R_P}{P_P}} \frac{M_P}{P_P M_P} \leq 1$, and by our assumption M_P is a finitely generated R_P-module, therefore by [23, Theorem 2.3], M_P is a cyclic R_P-module.

If $rank_{\frac{R_P}{P_P}} \frac{M_P}{P_P M_P} > 1$, then there exists a chain of submodules $\frac{N_1}{P_P M_P} \subset \frac{N_2}{P_P M_P} \subset \frac{M_P}{P_P M_P}$ of $\frac{M_P}{P_P M_P}$. So $P_P M_P \subseteq N_1 \subset N_2 \subset M_P$, and then $P_P \subseteq (P_P M_P : M_P) \subseteq (N_1 : M_P) \subseteq (N_2 : M_P) \subset R_P$. We know that P_P is a maximal ideal of R_P, then $P_P = (N_1 : M_P) = (N_2 : M_P)$. Since P_P is a maximal ideal of R_P, so $\frac{M_P}{N_1}$ is a non-zero vector space over the field $\frac{R_P}{P_P}$, and thus $\frac{M_P}{N_1}$ is a torsion-free $\frac{R_P}{P_P}$-module. So N_1 is a P_P-prime submodule of M_P. Similarly N_2 is a P_P-prime submodule of M_P. By 4.1, $N_1 \cap M$ and $N_2 \cap M$ are P-prime submodules of M, and since $N_1 \subset N_2$, so $N_1 \cap M \subseteq N_2 \cap M$.

Now by our assumption we have $N_1 \cap M = N_2 \cap M$. Again by 4.1, $N_1 = (N_1 \cap M)_P = (N_2 \cap M)_P = N_2$, which is a contradiction.

Case 2. The ideal P is an arbitrary prime ideal of R.

Then P is contained in a maximal ideal $\mathfrak{M}$ of R. By Case 1, the $R_{\mathfrak{M}}$-module $M_{\mathfrak{M}}$ is cyclic, and therefore $(M_{\mathfrak{M}})_{P_{\mathfrak{M}}}$ is a cyclic $(R_{\mathfrak{M}})_{P_{\mathfrak{M}}}$-module. One can easily see that $(R_{\mathfrak{M}})_{P_{\mathfrak{M}}} \cong R_P$ and $(M_{\mathfrak{M}})_{P_{\mathfrak{M}}} \cong M_P$ as R_P-modules, hence M_P is a cyclic R_P-module.

(iii) $\Longrightarrow$ **(i)** Let P be a prime ideal of R. Suppose N_1 and N_2 are two P-prime submodules of M with $N_1 \subseteq N_2$. We show that $N_1 = N_2$. By 4.1, $(N_1)_P$ and $(N_2)_P$ are P_P-prime submodule of M_P and $(N_1)_P \subseteq (N_2)_P$. So $P_P M_P \subseteq (N_1)_P \subseteq (N_2)_P \subset M_P$, and then $\frac{(N_1)_P}{P_P M_P} \subseteq \frac{(N_2)_P}{P_P M_P} \subset \frac{M_P}{P_P M_P}$. Since M_P is a cyclic R_P-module, so by [23, Theorem 2.3], $rank_{\frac{R_P}{P_P}} \frac{M_P}{P_P M_P} \leq 1$. Hence $\frac{(N_1)_P}{P_P M_P} = \frac{(N_2)_P}{P_P M_P}$, and consequently $(N_1)_P = (N_2)_P$. Now by 4.1, $N_1 = (N_1)_P \cap M = (N_2)_P \cap M = N_2$. So M is a semi multiplication R-module. □

Corollary 4.3.

(i) *Every multiplication module is semi multiplication.*

(ii) *Every finitely generated semi multiplication module is multiplication.*

(iii) *Every semi multiplication module over a semi local ring is cyclic.*

(iv) *Every semi multiplication module over a local ring is hollow.*

Proof. **(i)** Let M be a multiplication R-module. Then by 2.1**(ii)**, for every prime ideal P of R, M_P is a cyclic R_P-module. So by 4.2, M is a semi multiplication R-module.

(ii) Let M be a finitely generated semi multiplication R-module. By 4.2, for every prime ideal P of R, M_P is a cyclic R_P-module. Thus for every prime ideal P of R, M_P is a multiplication R_P-module, and by 2.1**(iii)**, M is a multiplication R-module.

(iii) Let M be a semi multiplication R-module where R is a semi local ring. By 4.2, for each prime ideal P of R, M_P is a cyclic R_P-module. Now by 2.1**(v)**, M is cyclic.

(iv) The proof is obvious by part **(iii)**, and 2.2**(iv)**. □

Corollary 4.4. *Let M be a multiplication R-module. Then for every prime ideal P of R, M_P is a hollow R_P-module.*

Proof. By 2.1**(ii)**, for every prime ideal P of R, M_P is a cyclic R_P-module. If $M_P = 0$, then obviously M_P is a hollow module. If $M_P \neq 0$, since M_P is a non-zero cyclic R_P-module, so we have $Max(M) \neq \emptyset$, and obviously $Ann\ M_P \neq R_P$, and R_P is a local ring, consequently $\frac{R_P}{Ann\ M_P}$ is a local ring. Now by 2.2**(iv)**, M_P is a hollow R_P-module. □

Remark.

(i) Let $\Pi =$ The set of prime numbers, and $M = \oplus_{p\in\Pi} Z_p$, then obviously M is a Z-module, and for each $q \in \Pi$, we have $M_{(q)} = Z_q$, and $M_{(0)} = 0$. By 4.2, M is a semi multiplication module. It is easy to see that $Ann\ Z_2 = 2Z$, and $Ann\ \hat{Z}_2 = 0$, then by 2.1**(iv)**, M is not a multiplication Z-module.

(ii) If for every prime ideal P of R, M_P is a hollow R_P-module, then M is not necessarily a hollow R-module. For instance if $0 \neq R$ is a ring which is not local, then by 2.2**(ii)**, R as an R-module is not a hollow module. But for every prime ideal P of R, R_P is a non-zero local and cyclic R_P-module, so by 2.2 **(ii)**, R_P is a hollow R_P-module.

Proposition 4.5. *Let M be an R-module and let $\frac{R}{Ann\ M}$ be a local ring. Then the following are equivalent:*

(i) *M is a hollow R-module and $Max(M) \neq \emptyset$;*

(ii) *M_P is a hollow R_P-module for each prime ideal P of R;*

(iii) *M_P is a hollow R_P-module for each maximal ideal P of R.*

Proof. Since $\frac{R}{Ann\ M}$ is a local ring, so $M \neq 0$.

(i) $\Longrightarrow$ **(ii)** By 2.2**(v)**, M is multiplication. Now we have the result by 4.4.

(ii) $\Longrightarrow$ **(iii)** The proof is clear.

(iii) $\Longrightarrow$ **(i)** By 2.1**(v)**, M is a cyclic R-module and by 2.2**(iv)**, M is a hollow module, and $Max(M) \neq \emptyset$. □

Let M be a hollow R-module and let N be a submodule of M. It is easy to see that $\frac{M}{N}$ is a hollow R-module. In the following parts of this section we will characterize module M such that every submodule of M is a hollow module.

Recall that a module M is said to be a serial module if every two submodules of M are comparable with respect to inclusion.

Theorem 4.6. *Let M be an R-module. Then the following are equivalent:*

(i) *M is a serial module;*

(ii) *Every submodule of M is hollow;*

(iii) *Every submodule of M generated by two elements is hollow.*

Proof. **(i)** $\Longrightarrow$ **(ii)** Let N be a submodule of M such that $Max(N) \neq \emptyset$, and $N_1 + N_2 = N$ where N_1 is a maximal submodule of the R-module N and N_2 is a submodule of N. Since N_1 and N_2 are submodules of M and M is a serial module, so $N_1 \leq N_2$ or $N_2 \leq N_1$. $N_1 \leq N_2$ implies that $N = N_1 + N_2 = N_2$ which proves the case. If $N_2 \leq N_1$, then $N = N_1 + N_2 = N_1$ which is impossible, since N_1 is a maximal submodule of N.

(ii) $\Longrightarrow$ **(iii)** The proof is obvious.

(iii) $\Longrightarrow$ **(i)** Let M_1 and M_2 be submodules of M and let $M_1 \not\subseteq M_2$. Then there exists an element $x \in M_1 - M_2$. Let y be an arbitrary element of M_2. We show that $y \in M_1$. Let $N = Rx + Ry$. If $N = Ry$, then $Rx \subseteq Rx + Ry = N = Ry \subseteq M_2$. So $x \in M_2$ which is impossible. Hence Ry is a proper submodule of $N = Rx + Ry$, and since N is a finitely generated R-module, so there exists a maximal submodule N_0 of N containing Ry. Now we have

$$N = Rx + Ry \subseteq Rx + N_0 \subseteq N.$$

Thus $N = Rx + N_0$. By our assumption N is a hollow R-module, so $N = Rx$. Therefore,

$$Ry \subseteq Rx + Ry = N = Rx \subseteq M_1,$$

which implies that $y \in M_1$. □

Recall that a ring R is said to be an arithmetical ring, if for each ideals $a,\ b$ and c of R, we have, $a + (b \cap c) = (a + b) \cap (a + c)$ (see [17] or [18]).

Lemma 4.7. *[18, Theorem 1] A ring R is arithmetical if and only if for each prime (or maximal) ideal P of R, every two ideals of the ring R_P are comparable.*

Theorem 4.8. *Let M be a multiplication R-module. Consider the following statements:*

(i) *$\frac{R}{Ann\ M}$ is an arithmetical ring;*

(ii) *For each prime (or maximal) ideal P of R, every submodule of the R_P-module M_P is hollow;*

(iii) *For each prime (or maximal) ideal P of R, $M_P \neq 0$, and every submodule of the R_P-module M_P is hollow.*

Then **(i)** $\Longrightarrow$ **(ii)** *and* **(iii)** $\Longrightarrow$ **(i)**.

Proof. **(i)** $\Longrightarrow$ **(ii)** By 4.6, it is enough to show that M_P is a serial module. Let N_1 and N_2 be submodules of M_P. Since M is a multiplication R-module, it is easy to see that M is a multiplication $\frac{R}{Ann\ M}$-module. So by 2.1(**ii**), M_P is a multiplication $(\frac{R}{Ann\ M})_P$-module. Hence $N_1 = IM_P$, and $N_2 = JM_P$, for some ideals I and J of $(\frac{R}{Ann\ M})_P$. By 4.7, every two ideals of $(\frac{R}{Ann\ M})_P$ are comparable, thus $I \subseteq J$ or $J \subseteq I$, which implies that $N_1 \subseteq N_2$ or $N_2 \subseteq N_1$.

(iii) $\Longrightarrow$ **(i)** By 2.1(**ii**), $(Ann\ M)_P = (0 : M)_P = (0_P : M_P) = Ann\ M_P$. Hence $(\frac{R}{Ann\ M})_P \cong \frac{R_P}{(Ann\ M)_P} = \frac{R_P}{Ann\ M_P}$. Then by 4.7, it is enough to show that every two ideals of $\frac{R_P}{Ann\ M_P}$ are comparable.

Let $\frac{I}{Ann\ M_P}$ and $\frac{J}{Ann\ M_P}$ be two ideals of $\frac{R_P}{Ann\ M_P}$. By 4.6, M_P is a serial module, then let $IM_P \subseteq JM_P$. By 2.1(**ii**), M_P is a cyclic R_P-module, then put $M_P = R_P x$. For each $i \in I$, we have $ix \in IM_p \subseteq JM_P = Jx$. Then there exists a $j \in J$ such that $ix = jx$, i.e. $(i - j)x = 0$. That is, $i - j \in Ann\ M_P \subseteq J$. So $i - j \in J$, and consequently $i \in J$. Thus $I \subseteq J$, and then $\frac{I}{Ann\ M_P} \subseteq \frac{J}{Ann\ M_P}$. □

Corollary 4.9. *If M is a finitely generated multiplication R-module, then the following are equivalent:*

(i) $\frac{R}{Ann\ M}$ *is an arithmetical ring;*

(ii) *For each prime (or maximal) ideal P of R, every submodule of the R_P-module M_P is hollow.*

Proof. **(i) $\Longrightarrow$ (ii)** The proof is obvious by 4.8.

(ii) $\Longrightarrow$ (i) Since M is a finitely generated R-module, so $(Ann\ M)_P = (0 : M)_P = (0 : M_P) = Ann\ M_P$. Now follow the proof of 4.8, part **(iii) $\Longrightarrow$ (i)**. □

Lemma 4.10. *Let R be an integral domain and let M be an R-module. Consider the following statements:*

(i) *For each prime ideal P of R, $M_P \neq 0$;*

(ii) $Ann\ M = 0$;

(iii) *M is a non-torsion R-module.*

Then **(i) $\Longrightarrow$ (ii)** *and* **(iii) $\Longrightarrow$ (i)** *and if M is finitely generated, then* **(ii) $\Longrightarrow$ (iii)**.

Proof. **(i) $\Longrightarrow$ (ii)** Suppose $0 \neq r \in Ann\ M$. Put $S = \{r^n |\ n \in N\}\{0\}$. Obviously S is a multiplicatively closed subset of R, then there exists a prime ideal P of R such that $P \cap S = \emptyset$. So $r \in R - P$. Now let $\frac{m}{s}$ be an arbitrary element of M_P where $m \in M$, and $s \in R - P$. We have $r \in Ann\ M$, then $rm = 0$, and $r \in R - P$, this means $\frac{m}{s} = \frac{0}{1}$ in M_P. Hence $M_P = 0$.

(iii) $\Longrightarrow$ (i) Let P be a prime ideal of R such that $M_P = 0$, and let x be an arbitrary element of M. Then $\frac{x}{1} \in M_P = 0$, so there exists an element $r \in R - P$ such that $rx = 0$ which means x is a torsion element of M. So M is a torsion R-module.

(ii) $\Longrightarrow$ (iii) Let M be generated by $x_1,\ x_2,\ \cdots x_n$. If M is a torsion R-module. Then for each x_i there exists a non-zero element $r_i \in R$, such that $r_i x_i = 0$. Then obviously $0 \neq r_1 r_2 \cdots r_n \in Ann\ M$. □

Note. In [3, Proposition 2.4], we proved that every multiplication module over an integral domain is either torsion or torsion-free. So every non-torsion multiplication module over an integral domain is torsion-free.

Corollary 4.11. *Let R be an integral domain and let M be a non-torsion (torsion-free) multiplication R-module. Then the following are equivalent:*

(i) *R is an arithmetical ring;*

(ii) *For each prime (or maximal) ideal P of R, every submodule of the R_P-module M_P is hollow.*

Proof. Since M is non-torsion, so obviously $Ann\ M = 0$.

(i) $\Longrightarrow$ (ii) The proof is given by 4.8.

(ii) $\Longrightarrow$ (i) By part **(iii) $\Longrightarrow$ (i)** of 4.10 for each prime (or maximal) ideal P of R, $M_P \neq 0$. Now by 4.8, we have the result. □

Proposition 4.12. *Let M be a non-torsion R-module and $Max(M) \neq \emptyset$. Then the following are equivalent:*

(i) *Every submodule of M is hollow;*

(ii) *M is a multiplication R-module and every two ideals of R are comparable.*

Proof. **(i) $\Longrightarrow$ (ii)** By 2.2, M is multiplication. Let a be a non-torsion element of M, and let I_1 and I_2 be two ideals of R.

By 4.6, $I_1 a \subseteq I_2 a$, or $I_2 a \subseteq I_1 a$. Let $I_1 a \subseteq I_2 a$. So for each $i_1 \in I_1$, $i_1 a = i_2 a$ for some $i_2 \in I_2$. Then $(i_1 - i_2)a = 0$, and since a is not a torsion element, $i_1 = i_2$, that is $I_1 \subseteq I_2$.

(ii) $\Longrightarrow$ (i) Let N_1 and N_2 be submodules of M. Then $N_1 = IM$ and $N_2 = JM$ where I and J are ideals of M. $I \subseteq J$ implies that $N_1 \subseteq N_2$, and $J \subseteq I$ implies that $N_2 \subseteq N_1$. Hence by 4.6, every submodule of M is hollow. □

Theorem 4.13. *Let R be a Noetherian ring and let M be a hollow R-module such that $Max(M) \neq \emptyset$. Then the following are equivalent:*

(i) $\frac{R}{Ann\ M}$ *is an arithmetical ring;*

(ii) *Every submodule of M is a hollow R-module;*

(iii) *Every two ideals of R containing $Ann\ M$ are comparable.*

Proof. **(i) $\Longrightarrow$ (ii)** Since $Max(M) \neq \emptyset$, so $0 \neq M$. By 2.2, M is a finitely generated multiplication R-module. Let $0 \neq N$ be an arbitrary submodule of M. By 4.9, for each prime ideal P of R, N_P is a hollow R_P-module. Since M is a finitely generated R-module and R is a Noetherian ring, so M is a Noetherian module, and so N is a finitely generated R-module. By 2.2, $\frac{R}{Ann\ M}$ is a local ring and obviously $Ann\ M \subseteq Ann\ N \subset R$, then $\frac{R}{Ann\ N}$ is a local ring. Now by 4.5, N is a hollow R-module.

(ii) $\Longrightarrow$ (iii) Since every submodule of M is a hollow R-module, so it is easy to see that every submodule of M as an $\frac{R}{Ann\ M}$-module is a hollow $\frac{R}{Ann\ M}$-module. M is a finitely generated faithful $\frac{R}{Ann\ M}$-module, then by 4.10, M is a non-torsion $\frac{R}{Ann\ M}$-module. Now by 4.12, every two ideals of R containing $Ann\ M$ are comparable.

(iii) $\Longrightarrow$ (i) The proof is clear. □

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