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GENERALIZED BULLEN TYPE INEQUALITIES FOR LOCAL FRACTIONAL INTEGRALS AND ITS APPLICATIONS

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Abstract In this paper, we establish the generalized Bullen type inequalities involving local fractional integrals on fractal sets R^{α} $(0 < \alpha \leq 1)$ of real line numbers. Some applications of these inequalities in numerical integration and for special means are given.

1 Introduction

The classical Hermite-Hadamard inequality which was first published in [7] gives us an estimate of the mean value of a convex function $f : I \to \mathbb{R}$,

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

An account the history of this inequality can be found in [3]. Surveys on various generalizations and developments can be found in [11]. Recently in [4], the author established this inequality for twice differentiable functions. In the case where f is convex then there exists an estimation better than (1.1). For more information recent developments to above inequalities, please refer to *[3]-[6], [8], [9], [13]* and so on.

In [1], Bullen proved the following inequality which is known as Bullen's inequality for convex function:

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$. The inequality

$$\frac{1}{b-a}\int_a^b f(x)dx \leq \frac{1}{2}\left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2}\right].$$

2 Preliminaries

Recall the set R^{α} of real line numbers and use the Gao-Yang-Kang's idea to describe the definition of the local fractional derivative and local fractional integral, see [19, 20] and so on.

Recently, the theory of Yang's fractional sets [19] was introduced as follows.

For $0 < \alpha \leq 1$, we have the following α-type set of element sets:

Z^{α} : The α-type set of integer is defined as the set $\{0^{\alpha}, \pm 1^{\alpha}, \pm 2^{\alpha}, ..., \pm n^{\alpha}, ...\}$.

Q^{α} : The α-type set of the rational numbers is defined as the set $\{m^{\alpha} = \left(\frac{p}{q}\right)^{\alpha} : p, q \in Z, q \neq 0\}$.

J^{α} : The α-type set of the irrational numbers is defined as the set $\{m^{\alpha} \neq \left(\frac{p}{q}\right)^{\alpha} : p, q \in Z, q \neq 0\}$.

R^{α} : The α-type set of the real line numbers is defined as the set $R^{\alpha} = Q^{\alpha} \cup J^{\alpha}$.

If a^{α}, b^{α} and c^{α} belongs the set R^{α} of real line numbers, then

(1) $a^{\alpha} + b^{\alpha}$ and $a^{\alpha}b^{\alpha}$ belongs the set R^{α};

(2) $a^{\alpha} + b^{\alpha} = b^{\alpha} + a^{\alpha} = (a+b)^{\alpha} = (b+a)^{\alpha}$;

(3) $a^{\alpha}+(b^{\alpha}+c^{\alpha})=(a+b)^{\alpha}+c^{\alpha}$;
(4) $a^{\alpha}b^{\alpha}=b^{\alpha}a^{\alpha}=(ab)^{\alpha}=(ba)^{\alpha}$;
(5) $a^{\alpha}(b^{\alpha}c^{\alpha})=(a^{\alpha}b^{\alpha})c^{\alpha}$;
(6) $a^{\alpha}(b^{\alpha}+c^{\alpha})=a^{\alpha}b^{\alpha}+a^{\alpha}c^{\alpha}$;
(7) $a^{\alpha}+0^{\alpha}=0^{\alpha}+a^{\alpha}=a^{\alpha}$ and $a^{\alpha}1^{\alpha}=1^{\alpha}a^{\alpha}=a^{\alpha}$.

The definition of the local fractional derivative and local fractional integral can be given as follows.

Definition 2.1. [19] A non-differentiable function $f:R\rightarrow R^{\alpha}$, $x\rightarrow f(x)$ is called to be local fractional continuous at x_0, if for any $\varepsilon>0$, there exists $\delta>0$, such that

$$|f(x)-f(x_0)|<\varepsilon^{\alpha}$$

holds for $|x-x_0|<\delta$, where $\varepsilon,\delta\in R$. If $f(x)$ is local continuous on the interval (a,b), we denote $f(x)\in C_{\alpha}(a,b)$.

Definition 2.2. [19] The local fractional derivative of $f(x)$ of order α at $x=x_0$ is defined by

$$f^{(\alpha)}(x_0)=\left.\frac{d^{\alpha}f(x)}{dx^{\alpha}}\right|_{x=x_0}=\lim_{x\rightarrow x_0}\frac{\Delta^{\alpha}(f(x)-f(x_0))}{(x-x_0)^{\alpha}},$$

where $\Delta^{\alpha}(f(x)-f(x_0))\cong\Gamma(\alpha+1)(f(x)-f(x_0))$.

If there exists $f^{(k+1)\alpha}(x)=\overbrace{D_x^{\alpha}...D_x^{\alpha}}^{k+1\text{ times}}f(x)$ for any $x\in I\subseteq R$, then we denoted $f\in D_{(k+1)\alpha}(I)$, where $k=0,1,2,...$

Definition 2.3. [19] Let $f(x)\in C_{\alpha}[a,b]$. Then the local fractional integral is defined by,

$${}_aI_b^{\alpha}f(x)=\frac{1}{\Gamma(\alpha+1)}\int_a^b f(t)(dt)^{\alpha}=\frac{1}{\Gamma(\alpha+1)}\lim_{\Delta t\rightarrow 0}\sum_{j=0}^{N-1}f(t_j)(\Delta t_j)^{\alpha},$$

with $\Delta t_j=t_{j+1}-t_j$ and $\Delta t=\max\{\Delta t_1,\Delta t_2,...,\Delta t_{N-1}\}$, where $[t_j,t_{j+1}]$, $j=0,...,N-1$ and $a=t_0<t_1<...<t_{N-1}<t_N=b$ is partition of interval $[a,b]$.

Here, it follows that ${}_aI_b^{\alpha}f(x)=0$ if $a=b$ and ${}_aI_b^{\alpha}f(x)=-{}_bI_a^{\alpha}f(x)$ if $a<b$. If for any $x\in[a,b]$, there exists ${}_aI_x^{\alpha}f(x)$, then we denoted by $f(x)\in I_x^{\alpha}[a,b]$.

Definition 2.4 (Generalized convex function). [19] Let $f:I\subseteq R\rightarrow R^{\alpha}$. For any $x_1,x_2\in I$ and $\lambda\in[0,1]$, if the following inequality

$$f(\lambda x_1+(1-\lambda)x_2)\leq\lambda^{\alpha}f(x_1)+(1-\lambda)^{\alpha}f(x_2)$$

holds, then f is called a generalized convex function on I.

Here are two basic examples of generalized convex functions:
(1) $f(x)=x^{\alpha p}$, $x\geq 0$, $p>1$;
(2) $f(x)=E_{\alpha}(x^{\alpha})$, $x\in R$ where $E_{\alpha}(x^{\alpha})=\sum\limits_{k=0}^{\infty}\frac{x^{\alpha k}}{\Gamma(1+k\alpha)}$ is the Mittag-Lrffer function.

Theorem 2.5. *[12] Let $f\in D_{\alpha}(I)$, then the following conditions are equivalent*
a) f is a generalized convex function on I
b) $f^{(\alpha)}$ is an increasing function on I
c) for any $x_1,x_2\in I$,

$$f(x_2)-f(x_1)\geq\frac{f^{(\alpha)}(x_1)}{\Gamma(1+\alpha)}(x_2-x_1)^{\alpha}.$$

Corollary 2.6. *[12] Let $f\in D_{2\alpha}(a,b)$. Then f is a generalized convex function (or a generalized concave function) if and only if*

$$f^{(2\alpha)}(x)\geq 0\left(or\ f^{(2\alpha)}(x)\leq 0\right)$$

for all $x\in(a,b)$.

Lemma 2.7. *[19]*

(1) (Local fractional integration is anti-differentiation) Suppose that $f(x) = g^{(\alpha)}(x) \in C_\alpha[a,b]$*, then we have*

$$_aI_b^\alpha f(x) = g(b) - g(a).$$

(2) (Local fractional integration by parts) Suppose that $f(x), g(x) \in D_\alpha[a,b]$ *and* $f^{(\alpha)}(x)$, $g^{(\alpha)}(x) \in C_\alpha[a,b]$*, then we have*

$$_aI_b^\alpha f(x)g^{(\alpha)}(x) = f(x)g(x)|_a^b - {_aI_b^\alpha} f^{(\alpha)}(x)g(x).$$

Lemma 2.8. *[19] We have*

i) $\dfrac{d^\alpha x^{k\alpha}}{dx^\alpha} = \dfrac{\Gamma(1+k\alpha)}{\Gamma(1+(k-1)\alpha)} x^{(k-1)\alpha}$;

ii) $\dfrac{1}{\Gamma(\alpha+1)} \int\limits_a^b x^{k\alpha}(dx)^\alpha = \dfrac{\Gamma(1+k\alpha)}{\Gamma(1+(k+1)\alpha)} \left(b^{(k+1)\alpha} - a^{(k+1)\alpha}\right), k \in R.$

Lemma 2.9 (Generalized Hölder's inequality). *[19] Let* $f, g \in C_\alpha[a,b]$, $p, q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$, *then*

$$\frac{1}{\Gamma(\alpha+1)} \int\limits_a^b |f(x)g(x)|\,(dx)^\alpha \leq \left(\frac{1}{\Gamma(\alpha+1)} \int\limits_a^b |f(x)|^p\,(dx)^\alpha\right)^{\frac{1}{p}} \left(\frac{1}{\Gamma(\alpha+1)} \int\limits_a^b |g(x)|^q\,(dx)^\alpha\right)^{\frac{1}{q}}.$$

In [12], Mo et al. proved the following generalized Hermite-Hadamard inequality for generalized convex function:

Theorem 2.10 (Generalized Hermite-Hadamard inequality). *Let* $f(x) \in I_x^{(\alpha)}[a,b]$ *be a generalized convex function on* $[a,b]$ *with* $a < b$. *Then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(1+\alpha)}{(b-a)^\alpha} {_aI_b^\alpha} f(x) \leq \frac{f(a)+f(b)}{2^\alpha}. \tag{2.1}$$

In [17], Sarikaya et al. proved the following generalized Bullen inequality for generalized convex function and they also established a equality involving local fractional integral with regard to generalized Bullen inequality.

Theorem 2.11 (Generalized Bullen inequality). *Let* $f(x) \in I_x^{(\alpha)}[a,b]$ *be a generalized convex function on* $[a,b]$ *with* $a < b$. *Then we have the inequality*

$$\frac{\Gamma(1+\alpha)}{(b-a)^\alpha} {_aI_b^\alpha} f(x) \leq \frac{1}{2^\alpha}\left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2^\alpha}\right].$$

Theorem 2.12. *Let* $I \subseteq \mathbb{R}$ *be an interval,* $f : I^0 \subseteq \mathbb{R} \to \mathbb{R}^\alpha$ *(*I^0 *is the interior of* I*) such that* $f \in D_{2\alpha}(I^0)$ *and* $f^{(2\alpha)} \in C_{2\alpha}[a,b]$ *for* $a, b \in I^0$ *with* $a < b$. *Then, for all* $x \in [a,b]$*, we have the identity*

$$\frac{1}{2^\alpha(b-a)^\alpha(\Gamma(1+\alpha))^2} \int\limits_a^b \left(x - \frac{a+b}{2}\right)^\alpha p(x) f^{(2\alpha)}(x)\,(dx)^\alpha \tag{2.2}$$

$$= \frac{1}{2^\alpha}\left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2^\alpha}\right] - \frac{\Gamma(1+\alpha)}{(b-a)^\alpha} {_aI_b^\alpha} f(x)$$

where

$$p(x) = \begin{cases} (a-x)^\alpha, & \left[a, \frac{a+b}{2}\right) \\ (b-x)^\alpha, & \left[\frac{a+b}{2}, b\right]. \end{cases}$$

The interested reader is able to look over the references [2], [12], [14]-[23] for local freactional theory.

In this study, firstly the generalized Bullen type inequalities are established. Then, some applications of these inequalities in numerical integration and for special means are given.

3 Main Results

In this section, we prove some inequalities which are generalized Bullen type inequalities involving local fractional integral.

Theorem 3.1. *Let $f(x) \in I_x^{(\alpha)}[a,b]$ be a generalized convex function on $[a,b]$ with $a<b$. Then the following inequality holds*

$$\frac{1}{2^\alpha}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2^\alpha}\right]-\frac{\Gamma(1+\alpha)}{(b-a)^\alpha}\,{}_aI_b^\alpha f(x)$$

$$\leq \frac{(b-a)^\alpha}{32^\alpha\Gamma(1+\alpha)}\left[f^{(\alpha)}(b)-f^{(\alpha)}(a)\right].$$

Proof. Since f is a generalized convex function, it follows that $f^{(2\alpha)} \geq 0$, for every $x \in [a,b]$. Because

$$0 \leq \left(x-\frac{a+b}{2}\right)^\alpha (a-x)^\alpha \leq \frac{(b-a)^{2\alpha}}{16^\alpha}$$

for any $x \in \left[a, \frac{a+b}{2}\right]$ and

$$0 \leq \left(x-\frac{a+b}{2}\right)^\alpha (b-x)^\alpha \leq \frac{(b-a)^{2\alpha}}{16^\alpha}$$

for any $x \in \left[\frac{a+b}{2}, b\right]$, we deduce the inequality

$$\left(x-\frac{a+b}{2}\right)^\alpha p(x) f^{(2\alpha)}(x) \leq \frac{(b-a)^{2\alpha}}{16^\alpha} f^{(2\alpha)}(x). \tag{3.1}$$

Integrating both sides of (3.1) with respect to x from a to b and using Lemma 2.7, we have

$$\frac{1}{\Gamma(1+\alpha)}\int_a^b \left(x-\frac{a+b}{2}\right)^\alpha p(x) f^{(2\alpha)}(x)(dx)^\alpha$$

$$\leq \frac{(b-a)^{2\alpha}}{16^\alpha}\frac{1}{\Gamma(1+\alpha)}\int_a^b f^{(2\alpha)}(x)(dx)^\alpha$$

$$= \frac{(b-a)^{2\alpha}}{16^\alpha}\left[f^{(\alpha)}(b)-f^{(\alpha)}(a)\right].$$

Using equality (3.1) in the previous inequality, we easily find required inequality. □

Remark 3.2. If we choose $\alpha = 1$ in Theorem 3.1, then we have the following inequality

$$\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]-\frac{1}{b-a}\int_a^b f(x)dx$$

$$\leq \frac{b-a}{32}\left[f'(b)-f'(a)\right]$$

which is proved by Minculate et all. in [10].

Theorem 3.3. *We suppose again that the assumptions of Theorem 2.12 are satisfied. If $f^{(2\alpha)}$ is bounded on (a,b), then we have the inequality*

$$\left|\frac{1}{2^\alpha}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2^\alpha}\right]-\frac{\Gamma(1+\alpha)}{(b-a)^\alpha}\,{}_aI_b^\alpha f(x)\right| \tag{3.2}$$

$$\leq \frac{(b-a)^{2\alpha}}{8^\alpha\Gamma(1+\alpha)}\left[\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}-\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right]\left\|f^{(2\alpha)}\right\|_\infty.$$

Proof. Taking madulus in (2.2) and using bounded of $f^{(2\alpha)}$, we find that

$$\left|\frac{1}{2^{\alpha}}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2^{\alpha}}\right]-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\,{}_{a}I_{b}^{\alpha}f(x)\right| \tag{3.3}$$

$$\begin{aligned}
&\le \frac{1}{2^{\alpha}(b-a)^{\alpha}(\Gamma(1+\alpha))^{2}}\int_{a}^{b}\left|x-\frac{a+b}{2}\right|^{\alpha}|p(x)|\left|f^{(2\alpha)}(x)\right|(dx)^{\alpha}\\
&\le \frac{\left\|f^{(2\alpha)}\right\|_{\infty}}{2^{\alpha}(b-a)^{\alpha}\Gamma(1+\alpha)}\frac{1}{\Gamma(1+\alpha)}\int_{a}^{b}\left|x-\frac{a+b}{2}\right|^{\alpha}|p(x)|(dx)^{\alpha}\\
&= \frac{\left\|f^{(2\alpha)}\right\|_{\infty}}{2^{\alpha}(b-a)^{\alpha}\Gamma(1+\alpha)}K.
\end{aligned}$$

Now, we calculate the integral K by using the Lemma 2.8, we have

$$\begin{aligned}
K &= \frac{1}{\Gamma(1+\alpha)}\int_{a}^{\frac{a+b}{2}}\left(\frac{a+b}{2}-x\right)^{\alpha}(x-a)^{\alpha}(dx)^{\alpha}\\
&\quad+\frac{1}{\Gamma(1+\alpha)}\int_{\frac{a+b}{2}}^{b}\left(x-\frac{a+b}{2}\right)^{\alpha}(b-x)^{\alpha}(dx)^{\alpha}.
\end{aligned}$$

Applying the change of the variables $x-a=u$ and $b-x=v$, we write

$$\begin{aligned}
K &= \frac{1}{\Gamma(1+\alpha)}\left[\int_{0}^{\frac{b-a}{2}}\left(\frac{b-a}{2}-u\right)^{\alpha}u^{\alpha}(du)^{\alpha}+\int_{0}^{\frac{b-a}{2}}\left(\frac{b-a}{2}-v\right)^{\alpha}v^{\alpha}(dv)^{\alpha}\right]\\
&= \frac{2^{\alpha}}{\Gamma(1+\alpha)}\int_{0}^{\frac{b-a}{2}}\left(\frac{b-a}{2}-u\right)^{\alpha}u^{\alpha}(du)^{\alpha}\\
&= \frac{(b-a)^{3\alpha}}{4^{\alpha}}\left[\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}-\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right].
\end{aligned}$$

If we substitute the integral K in (3.3), then we obtain desired result. □

Theorem 3.4. *We suppose again that the assumptions of Theorem 2.12 are satisfied. If $\left|f^{(2\alpha)}\right|$ is generalized convex, then we have the inequality*

$$\left|\frac{1}{2^{\alpha}}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2^{\alpha}}\right]-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\,{}_{a}I_{b}^{\alpha}f(x)\right| \tag{3.4}$$

$$\le \frac{(b-a)^{2\alpha}}{16^{\alpha}\Gamma(1+\alpha)}\left[\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}-\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right]\left[\left|f^{(2\alpha)}(a)\right|+\left|f^{(2\alpha)}(b)\right|\right].$$

Proof. Taking madulus in (2.2), we find that

$$\left|\frac{1}{2^{\alpha}}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2^{\alpha}}\right]-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\,{}_aI_b^{\alpha}f(x)\right| \tag{3.5}$$

$$\begin{aligned}
&\leq \frac{1}{2^{\alpha}(b-a)^{\alpha}(\Gamma(1+\alpha))^{2}}\int_a^b\left|x-\frac{a+b}{2}\right|^{\alpha}|p(x)|\left|f^{(2\alpha)}(x)\right|(dx)^{\alpha}\\
&= \frac{1}{2^{\alpha}(b-a)^{\alpha}\Gamma(1+\alpha)}\left[\frac{1}{\Gamma(1+\alpha)}\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-x\right)^{\alpha}(x-a)^{\alpha}\left|f^{(2\alpha)}(x)\right|(dx)^{\alpha}\right.\\
&\left.+\frac{1}{\Gamma(1+\alpha)}\int_{\frac{a+b}{2}}^{b}\left(x-\frac{a+b}{2}\right)^{\alpha}(b-x)^{\alpha}\left|f^{(2\alpha)}(x)\right|(dx)^{\alpha}\right]\\
&= \frac{1}{2^{\alpha}(b-a)^{\alpha}\Gamma(1+\alpha)}[I_1+I_2].
\end{aligned}$$

Since $\left|f^{(2\alpha)}\right|$ is generalized convex on $[a,b]$, we have

$$\begin{aligned}
\left|f^{(2\alpha)}(x)\right| &= \left|f^{(2\alpha)}\left(\frac{x-a}{b-a}b+\frac{b-x}{b-a}a\right)\right| \\
&\leq \left(\frac{x-a}{b-a}\right)^{\alpha}\left|f^{(2\alpha)}(b)\right|+\left(\frac{b-x}{b-a}\right)^{\alpha}\left|f^{(2\alpha)}(a)\right|.
\end{aligned} \tag{3.6}$$

Now, we calculate the integrals I_1 and I_2 by using of the inequality (3.6), we obtain

$$\begin{aligned}
I_1 &\leq \frac{\left|f^{(2\alpha)}(b)\right|}{\Gamma(1+\alpha)(b-a)^{\alpha}}\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-x\right)^{\alpha}(x-a)^{2\alpha}(dx)^{\alpha}\\
&+\frac{\left|f^{(2\alpha)}(a)\right|}{\Gamma(1+\alpha)(b-a)^{\alpha}}\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-x\right)^{\alpha}(x-a)^{\alpha}(b-x)^{\alpha}(dx)^{\alpha}.
\end{aligned}$$

If we write $(b-a-(x-a))^{\alpha}$ instead of $(b-x)^{\alpha}$ and also we use the change of the variable $x-a=u$, then we get

$$\begin{aligned}
I_1 &\leq \frac{\left|f^{(2\alpha)}(b)\right|}{(b-a)^{\alpha}\Gamma(1+\alpha)}\int_0^{\frac{b-a}{2}}\left(\frac{b-a}{2}-u\right)^{\alpha}u^{2\alpha}(du)^{\alpha}\\
&+\frac{\left|f^{(2\alpha)}(a)\right|}{\Gamma(1+\alpha)}\int_0^{\frac{b-a}{2}}\left(\frac{b-a}{2}-u\right)^{\alpha}u^{\alpha}(du)^{\alpha}\\
&-\frac{\left|f^{(2\alpha)}(a)\right|}{(b-a)^{\alpha}\Gamma(1+\alpha)}\int_0^{\frac{b-a}{2}}\left(\frac{b-a}{2}-u\right)^{\alpha}u^{2\alpha}(du)^{\alpha}.
\end{aligned}$$

Using Lemma 2.8, we have

$$\begin{aligned} I_1 &\leq \frac{\left|f^{(2\alpha)}(b)\right|}{(b-a)^{\alpha}}\left(\frac{b-a}{2}\right)^{4\alpha}\left[\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}-\frac{\Gamma(1+3\alpha)}{\Gamma(1+4\alpha)}\right] \\ &+\left|f^{(2\alpha)}(a)\right|\left(\frac{b-a}{2}\right)^{3\alpha}\left[\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}-\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right] \\ &-\frac{\left|f^{(2\alpha)}(a)\right|}{(b-a)^{\alpha}}\left(\frac{b-a}{2}\right)^{4\alpha}\left[\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}-\frac{\Gamma(1+3\alpha)}{\Gamma(1+4\alpha)}\right]. \end{aligned} \tag{3.7}$$

Similarly, writing $(b-a-(b-x))^{\alpha}$ instead of $(x-a)^{\alpha}$ and also using the change of the variable $b-x=v$, we obtain

$$\begin{aligned} I_2 &\leq \left|f^{(2\alpha)}(b)\right|\left(\frac{b-a}{2}\right)^{3\alpha}\left[\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}-\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right] \\ &-\frac{\left|f^{(2\alpha)}(b)\right|}{(b-a)^{\alpha}}\left(\frac{b-a}{2}\right)^{4\alpha}\left[\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}-\frac{\Gamma(1+3\alpha)}{\Gamma(1+4\alpha)}\right] \\ &+\frac{\left|f^{(2\alpha)}(a)\right|}{(b-a)^{\alpha}}\left(\frac{b-a}{2}\right)^{4\alpha}\left[\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}-\frac{\Gamma(1+3\alpha)}{\Gamma(1+4\alpha)}\right]. \end{aligned} \tag{3.8}$$

If we substitute the inequalities (3.7) and (3.8) in (3.5) and also we use elementary analysis, then we easily deduce desired inequality. □

Theorem 3.5. *We suppose again that the assumptions of Theorem 2.12 are satisfied. If* $\left|f^{(2\alpha)}\right|^{q}$ *is generalized convex, then we have the inequality*

$$\begin{aligned} &\left|\frac{1}{2^{\alpha}}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2^{\alpha}}\right]-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\,{}_aI_b^{\alpha}f(x)\right| \\ \leq\ &\frac{(b-a)^{2\alpha}}{8^{\alpha}(\Gamma(1+\alpha))^{\frac{1}{p}}(\Gamma(1+2\alpha))^{\frac{1}{q}}} \\ &\times\left[\left|f^{(2\alpha)}(a)\right|^{q}+\left|f^{(2\alpha)}(b)\right|^{q}\right]^{\frac{1}{q}}[B(p+1,p+1)]^{\frac{1}{p}} \end{aligned}$$

where, $p,q>1$, $\frac{1}{p}+\frac{1}{q}=1$, *and* B *is defined by*

$$B(x,y)=\frac{1}{\Gamma(1+\alpha)}\int_0^1 t^{(x-1)\alpha}(1-t)^{(y-1)\alpha}(dt)^{\alpha}.$$

Proof. Taking madulus in (2.2) and using generalized Hölder's inequality, we find that

$$\left|\frac{1}{2^{\alpha}}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2^{\alpha}}\right]-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_{a}I_{b}^{\alpha}f(x)\right| \tag{3.9}$$

$$\begin{aligned}
&\leq \frac{1}{2^{\alpha}(b-a)^{\alpha}(\Gamma(1+\alpha))^{2}}\int_{a}^{b}\left|x-\frac{a+b}{2}\right|^{\alpha}|p(x)|\left|f^{(2\alpha)}(x)\right|(dx)^{\alpha}\\
&\leq \frac{1}{2^{\alpha}(b-a)^{\alpha}\Gamma(1+\alpha)}\left(\frac{1}{\Gamma(1+\alpha)}\int_{a}^{b}\left|f^{(2\alpha)}(t)\right|^{q}(dt)^{\alpha}\right)^{\frac{1}{q}}\\
&\quad\times\left(\frac{1}{\Gamma(1+\alpha)}\int_{a}^{b}\left|x-\frac{a+b}{2}\right|^{\alpha p}|p(x)|^{p}(dt)^{\alpha}\right)^{\frac{1}{p}}\\
&= \frac{1}{2^{\alpha}(b-a)^{\alpha}\Gamma(1+\alpha)}\left(\frac{1}{\Gamma(1+\alpha)}\int_{a}^{b}\left|f^{(2\alpha)}(t)\right|^{q}(dt)^{\alpha}\right)^{\frac{1}{q}}(L)^{\frac{1}{p}}.
\end{aligned}$$

Now, we calculate the integral L by using the Lemma 2.8, we get

$$\begin{aligned}
L &= \frac{1}{\Gamma(1+\alpha)}\int_{a}^{\frac{a+b}{2}}\left(\frac{a+b}{2}-x\right)^{\alpha p}(x-a)^{\alpha p}(dx)^{\alpha}\\
&\quad+\frac{1}{\Gamma(1+\alpha)}\int_{\frac{a+b}{2}}^{b}\left(x-\frac{a+b}{2}\right)^{\alpha p}(b-x)^{\alpha p}(dx)^{\alpha}\\
&= L_{1}+L_{2}.
\end{aligned}$$

For calculating integral L_1, using changing variable with $x=(1-t)a+t\frac{a+b}{2}$, we obtain

$$\begin{aligned}
L_{1} &= \frac{1}{\Gamma(1+\alpha)}\int_{a}^{\frac{a+b}{2}}\left(\frac{a+b}{2}-x\right)^{p\alpha}(x-a)^{p\alpha}(dx)^{\alpha} \\
&= \left(\frac{b-a}{2}\right)^{(2p+1)\alpha}\frac{1}{\Gamma(1+\alpha)}\int_{0}^{1}(1-t)^{p\alpha}t^{p\alpha}(dt)^{\alpha}\\
&= \left(\frac{b-a}{2}\right)^{(2p+1)\alpha}B(p+1,p+1).
\end{aligned} \tag{3.10}$$

Similarliy, using changing variable with $x=(1-t)\frac{a+b}{2}+tb$, we have

$$\begin{aligned}
L_{2} &= \frac{1}{\Gamma(1+\alpha)}\int_{\frac{a+b}{2}}^{b}\left(x-\frac{a+b}{2}\right)^{p\alpha}(b-x)^{p\alpha}(dx)^{\alpha} \\
&= \left(\frac{b-a}{2}\right)^{(2p+1)\alpha}B(p+1,p+1).
\end{aligned} \tag{3.11}$$

Since $\left|f^{(2\alpha)}\right|$ is generalized convex on $[a,b]$, we have

$$\begin{aligned} \left|f^{(2\alpha)}(x)\right|^{q} &= \left|f^{(2\alpha)}\left(\frac{x-a}{b-a}b+\frac{b-x}{b-a}a\right)\right|^{q} \\ &\leq \left(\frac{x-a}{b-a}\right)^{\alpha}\left|f^{(2\alpha)}(b)\right|^{q}+\left(\frac{b-x}{b-a}\right)^{\alpha}\left|f^{(2\alpha)}(a)\right|^{q}. \end{aligned} \tag{3.12}$$

Using the inequality (3.12), we obtain

$$\frac{1}{\Gamma(1+\alpha)}\int_{a}^{b}\left|f^{(2\alpha)}(t)\right|^{q}(dt)^{\alpha} \tag{3.13}$$

$$\begin{aligned} &\leq \frac{\left|f^{(2\alpha)}(b)\right|^{q}}{\Gamma(1+\alpha)}\int_{a}^{b}\left(\frac{x-a}{b-a}\right)^{\alpha}(dt)^{\alpha}+\frac{\left|f^{(2\alpha)}(a)\right|^{q}}{\Gamma(1+\alpha)}\int_{a}^{b}\left(\frac{b-x}{b-a}\right)^{\alpha}(dt)^{\alpha} \\ &= \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}(b-a)^{\alpha}\left[\left|f^{(2\alpha)}(a)\right|^{q}+\left|f^{(2\alpha)}(b)\right|^{q}\right]. \end{aligned}$$

If we substitute (3.10), (3.11) and (3.13) in (3.9) and also we use elementary analysis, then we easily deduce desired inequality. □

4 Applications to Numerical Integration

We now consider applications of the integral inequalities involving local fractional integral developed in the previous section, to obtain estimates of composite quadrature rules which, it turns out have a markedly smaller error than that which may be obtained by the classical results.

Theorem 4.1. *Let $f:[a,b]\to\mathbb{R}^{\alpha}$ be $f\in D_{2\alpha}(a,b)$ and $f^{(2\alpha)}$ is bounded on (a,b). If $I_n: a=x_0<x_1<...<x_{n-1}<x_n=b$ is a partition of $[a,b]$ and $h_i=(x_{i+1}-x_i)$, $i=0,...,n-1$, then we have:*

$$\frac{1}{\Gamma(1+\alpha)}\int_{a}^{b}f(x)(dx)^{\alpha}=B(I_n,f)+R(I_n,f)$$

where

$$B(I_n,f)=\frac{1}{2^{\alpha}\Gamma(1+\alpha)}\left[\sum_{i=0}^{n-1}f\left(\frac{x_i+x_{i+1}}{2}\right)h_i^{\alpha}+\sum_{i=0}^{n-1}\frac{f(x_i)+f(x_{i+1})}{2^{\alpha}}h_i^{\alpha}\right]$$

and the remainder term satisfies the astimation:

$$|R(I_n,f)|\leq\frac{\left\|f^{(2\alpha)}\right\|_{\infty}}{8^{\alpha}\left(\Gamma(1+\alpha)\right)^{2}}\left[\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}-\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right]\sum_{i=0}^{n-1}h_i^{3\alpha}. \tag{4.1}$$

Proof. Applying Theorem 3.3 on the interval $[x_i,x_{i+1}]$, $i=0,...,n-1$, we obtain

$$\begin{aligned} &\left|\frac{1}{2^{\alpha}\Gamma(1+\alpha)}\left[f\left(\frac{x_i+x_{i+1}}{2}\right)h_i^{\alpha}+\frac{f(x_i)+f(x_{i+1})}{2^{\alpha}}h_i^{\alpha}\right]-{}_{x_i}I_{x_{i+1}}^{\alpha}f(x)\right| \\ &\leq \frac{\left\|f^{(2\alpha)}\right\|_{\infty}}{8^{\alpha}\left(\Gamma(1+\alpha)\right)^{2}}\left[\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}-\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right]h_i^{3\alpha} \end{aligned}$$

for all $i=0,...,n-1$. Summing over i from 0 to $n-1$ and using the triangle inequality we obtain the estimation (4.1). □

Theorem 4.2. *Let $B(I_n, f)$ and $R(I_n, f)$ be as defined in Theorem 4.1. If $\left|f^{(2\alpha)}\right|^q$ is a generalized convex function on $[a,b]$ and also $I_n : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ is a partition of $[a,b]$ and $h_i = (x_{i+1} - x_i)$, $i = 0, ..., n-1$, then we have:*

$$|R(I_n, f)| \tag{4.2}$$

$$\leq \frac{(B(p+1,p+1))^{\frac{1}{p}}}{8^\alpha (\Gamma(1+\alpha))^{1+\frac{1}{p}} (\Gamma(1+2\alpha))^{\frac{1}{q}}} \sum_{i=0}^{n-1} h_i^{3\alpha} \left[\left|f^{(2\alpha)}(x_i)\right|^q + \left|f^{(2\alpha)}(x_{i+1})\right|^q\right]^{\frac{1}{q}}$$

where, $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $B(x,y)$ is defined by

$$B(x,y) = \frac{1}{\Gamma(1+\alpha)} \int_0^1 t^{(x-1)\alpha} (1-t)^{(y-1)\alpha} (dt)^\alpha .$$

Proof. Applying Theorem 3.5 on the interval $[x_i, x_{i+1}]$, $i = 0, ..., n-1$, we obtain

$$\left|\frac{1}{2^\alpha \Gamma(1+\alpha)} \left[f\left(\frac{x_i + x_{i+1}}{2}\right) h_i^\alpha + \frac{f(x_i) + f(x_{i+1})}{2^\alpha} h_i^\alpha\right] - {}_{x_i}I_{x_{i+1}}^\alpha f(x)\right|$$

$$\leq \frac{(B(p+1,p+1))^{\frac{1}{p}}}{8^\alpha (\Gamma(1+\alpha))^{1+\frac{1}{p}} (\Gamma(1+2\alpha))^{\frac{1}{q}}} h_i^{3\alpha} \left[\left|f^{(2\alpha)}(x_i)\right|^q + \left|f^{(2\alpha)}(x_{i+1})\right|^q\right]^{\frac{1}{q}}$$

for all $i = 0, ..., n-1$. Summing over i from 0 to $n-1$ and using the triangle inequality we obtain the estimation (4.2) which completes the proof. □

5 Applications to Some Special Means

Let us recall some generalized means:

$$A(a,b) = \frac{a^\alpha + b^\alpha}{2^\alpha};$$

$$L_n(a,b) = \left[\frac{\Gamma(1+n\alpha)}{\Gamma(1+(n+1)\alpha)} \left[\frac{b^{(n+1)\alpha} - a^{(n+1)\alpha}}{(b-a)^\alpha}\right]\right]^{\frac{1}{n}}, \ n \in Z \setminus \{-1, 0\}, \ a, b \in R, \ a \neq b.$$

Now, let us reconsider the inequality (3.2):

$$\left|\frac{1}{2^\alpha}\left[f\left(\frac{a+b}{2}\right) + \frac{f(a) + f(b)}{2^\alpha}\right] - \frac{\Gamma(1+\alpha)}{(b-a)^\alpha} {}_aI_b^\alpha f(x)\right|$$

$$\leq \frac{(b-a)^{2\alpha}}{8^\alpha \Gamma(1+\alpha)} \left[\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} - \frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right] \left\|f^{(2\alpha)}\right\|_\infty$$

for all $x \in [a,b]$.

Consider the mapping $f : (0,\infty) \to R^\alpha$, $f(x) = x^{n\alpha}$, $n \in Z \setminus \{-1, 0\}$. Then, $0 < a < b$, we have

$$f\left(\frac{a+b}{2}\right) = [A(a,b)]^n, \quad \frac{f(a) + f(b)}{2^\alpha} = A(a^n, b^n)$$

and

$$\frac{1}{(b-a)^\alpha} {}_aI_b^\alpha f(t) = [L_n(a,b)]^n .$$

Using Lemma 2.8, we obtain

$$\left\|f^{(2\alpha)}\right\|_\infty = \begin{cases} \left|\frac{\Gamma(1+n\alpha)}{\Gamma(1+(n-2)\alpha)}\right| b^{(n-2)\alpha}, & n > 1 \\ \\ \left|\frac{\Gamma(1+n\alpha)}{\Gamma(1+(n-2)\alpha)}\right| a^{(n-2)\alpha}, & n \in (-\infty, 1] \setminus \{-1, 0\} \end{cases}$$

and then we deduce that

$$\left|\frac{1}{2^{\alpha}}\left[[A(a,b)]^{n}+A(a^{n},b^{n})\right]-\Gamma(1+\alpha)\left[L_{n}(a,b)\right]^{n}\right|$$

$$\leq \frac{(b-a)^{2\alpha}}{8^{\alpha}\Gamma(1+\alpha)}\left[\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}-\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right]\delta_{n}(a,b)$$

where

$$\delta_{n}(a,b)=\begin{cases}\left|\frac{\Gamma(1+n\alpha)}{\Gamma(1+(n-2)\alpha)}\right| b^{(n-2)\alpha}, & n>1\\ \\ \left|\frac{\Gamma(1+n\alpha)}{\Gamma(1+(n-2)\alpha)}\right| a^{(n-2)\alpha}, & n\in(-\infty,1]\setminus\{-1,0\}\end{cases}$$

for all $x\in[a,b]$.

Also, let $n>3$ for the function $f(x)=x^{n\alpha}$, $f:(0,\infty)\to R^{\alpha}$, then $\left|f^{(2\alpha)}\right|$ is a generalized convex function. Now, let us reconsider the inequality (3.4):

$$\left|\frac{1}{2^{\alpha}}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2^{\alpha}}\right]-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\,{}_{a}I_{b}^{\alpha}f(x)\right|$$

$$\leq \frac{(b-a)^{2\alpha}}{16^{\alpha}\Gamma(1+\alpha)}\left[\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}-\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right]\left[\left|f^{(2\alpha)}(a)\right|+\left|f^{(2\alpha)}(b)\right|\right].$$

Then, $0<a<b$, we have

$$\left|\frac{1}{2^{\alpha}}\left[[A(a,b)]^{n}+A(a^{n},b^{n})\right]-\Gamma(1+\alpha)\left[L_{n}(a,b)\right]^{n}\right|$$

$$\leq \frac{(b-a)^{2\alpha}}{16^{\alpha}\Gamma(1+\alpha)}\left[\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}-\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right]$$

$$\times\frac{\Gamma(1+n\alpha)}{\Gamma(1+(n-2)\alpha)}\left[a^{(n-2)\alpha}+b^{(n-2)\alpha}\right]$$

for all $x\in[a,b]$.

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